

# Complexity of the Description Logic $\mathcal{ALCM}$

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**Abstract.** In this paper we show that the problem of checking consistency of a knowledge base in the Description Logic  $\mathcal{ALCM}$  is ExpTime-complete. The  $\mathcal{M}$  stands for meta-modelling as defined by Motz, Rohrer and Severi. To show our main result, we define an ExpTime Tableau algorithm as an extension of an algorithm for checking consistency of a knowledge base in  $\mathcal{ALC}$  by Nguyen and Szalas.

## 1 Introduction

The main motivation of the present work is to study the complexity of meta-modelling as defined in [1,2]. No study of complexity has been done so far for this approach and we would like to analyse if it increases the complexity of a given description logic.

It is well-known that consistency of a (general) knowledge base in  $\mathcal{ALC}$  is ExpTime-complete. The hardness result was proved in [3]. A matching upper bound for  $\mathcal{ALC}$  was given by De Giacomo and Lenzerini by a reduction to PDL [4].

In this paper, we show that the consistency of a knowledge base in  $\mathcal{ALCM}$  is ExpTime-complete where the  $\mathcal{M}$  stands for the meta-modelling approach mentioned above. Hardness follows trivially from the fact that  $\mathcal{ALCM}$  is an extension of  $\mathcal{ALC}$  since any algorithm that decides consistency of a knowledge base in  $\mathcal{ALCM}$  can be used to decide consistency of a knowledge base in  $\mathcal{ALC}$ . In order to give a matching upper bound on the complexity of this problem, it is enough to show that there is a particular algorithm with running time at most  $O(2^n)$  where  $n$  is the size of the knowledge base. The standard tableau algorithm for  $\mathcal{ALC}$  which builds completion trees, e.g. see [5], can be extended with the expansion rules for meta-modelling of [1,2]. This algorithm is the bases for Semantic Web reasoners such as Pellet [6]. However, it has a high (worse case) complexity, namely NExpTime, and cannot be used to prove that the consistency problem for  $\mathcal{ALCM}$  is ExpTime-complete.

Other approaches to meta-modelling use translations to prove decidability and/or complexity [7,8,9,10,11,12,13,14]. However, translations do not seem to work for  $\mathcal{ALCM}$  due to the combination of a flexible syntax with a strong semantics of well-founded sets. A consistency algorithm for  $\mathcal{ALCM}$  has to check if the domain of the canonical model under construction is a well-founded set. This is an unusual and interesting aspect of our approach but at the same time what makes it more difficult to solve.

The contributions of this paper are the following:

1. We define a tableau algorithm for checking consistency of a knowledge base in  $\mathcal{ALCM}$  as an extension of an algorithm for  $\mathcal{ALC}$  by Nguyen and Szalas [15].

2. We prove correctness and show that the complexity of our algorithm for  $\mathcal{ALCCM}$  is ExpTime.
3. From the above two items, we obtain the main result of our paper which is the fact that the problem of checking consistency of a knowledge base in  $\mathcal{ALCCM}$  is ExpTime-complete.

Hence, in spite of the fact that our algorithm has the burden of having to check for well-founded sets, complexity does not change when moving from  $\mathcal{ALC}$  to  $\mathcal{ALCCM}$ .

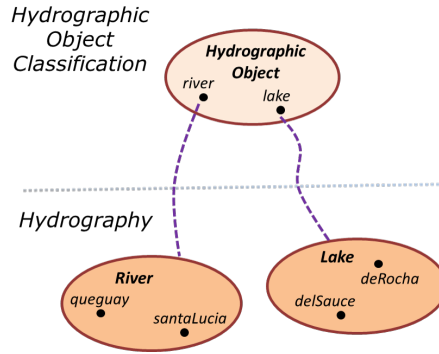
## 2 A Flexible Meta-modelling Approach for Re-using Ontologies

A knowledge base in  $\mathcal{ALCCM}$  contains an Mbox besides of a Tbox and an Abox. An Mbox is a set of equalities of the form  $a =_m A$  where  $a$  is an individual and  $A$  is a concept [1,2]. Figure 1 shows an example of two ontologies separated by a horizontal line. The two ontologies conceptualize the same entities at different levels of granularity. In the ontology above the horizontal line, rivers and lakes are formalized as individuals while in the one below the line they are concepts. If we want to integrate these ontologies into a single ontology (or into an ontology network) it is necessary to interpret the individual *river* and the concept *River* as the same real object. Similarly for *lake* and *Lake*. The Mbox for this example contains two equations:

$$river =_m River \quad lake =_m Lake$$

These equalities are called *meta-modelling axioms* and in this case, we say that the ontologies are related through *meta-modelling*. In Figure 1, meta-modelling axioms are represented by dashed edges. After adding the meta-modelling axioms for rivers and lakes, the concept *HydrographicObject* is now also a *meta-concept* because it is a concept that contains an individual which is also a concept.

This kind of meta-modelling can be expressed in OWL Full but it cannot be expressed



**Fig. 1.** Two ontologies on Hydrography

in OWL DL. The fact that it is expressed in OWL Full is not very useful since the meta-modelling provided by OWL Full is so expressive that leads to undecidability [7]. OWL

2 DL has a very restricted form of meta-modelling called *punning* where the same identifier can be used as an individual and as a concept [16]. These identifiers are treated as different objects by the reasoner and it is not possible to detect certain inconsistencies. We next illustrate two examples where OWL would not detect inconsistencies because the identifiers, though they look syntactically equal, are actually different.

*Example 1.* If we introduce an axiom expressing that *HydrographicObject* is a subclass of *River*, then OWL's reasoner will not detect that the interpretation of *River* is not a well founded set (it is a set that belongs to itself).

*Example 2.* We add two axioms, the first one says that *river* and *lake* as individuals are equal and the second one says that the classes *River* and *Lake* are disjoint. Then OWL's reasoner does not detect that there is a contradiction.

In order to detect these inconsistencies, *river* and *River* should be made semantically equal, i.e. the interpretations of the individual *river* and the concept *River* should be the same. The domain of an interpretation cannot longer consists of only basic objects but it must be any well-founded set. The well-foundedness of our model is not ensured by means of fixing layers beforehand as in [8,10] but it is the reasoner which checks for circularities. This approach allows the user to have any number of levels or layers (meta-concepts, meta meta-concepts and so on). The user does not have to write or know the layer of the concept because the reasoner will infer it for him. In this way, axioms can also naturally mix elements of different layers and the user has the flexibility of changing the status of an individual at any point without having to make any substantial change to the ontology.

### 3 The Description Logic $\mathcal{ALC}$

In this section we recall the Description Logic  $\mathcal{ALC}$  [17,5]. We assume we have three pairwise disjoint sets: a set of individuals, a set of atomic concepts and a set of atomic roles. Individuals are denoted by  $a, b, \dots$ , atomic concepts by  $A, B, \dots$  and atomic roles by  $R, S, \dots$ . We use  $C, D$  to denote arbitrary concepts. *Concepts* are defined by the following grammar:

$$C, D ::= A \mid \top \mid (\neg C) \mid (C \sqcap D) \mid (C \sqcup D) \mid (\forall R.C) \mid (\exists R.C)$$

We omit parenthesis according to the following precedence order of the description logics operators: (i)  $\neg, \forall, \exists$  (ii)  $\sqcap$ , (iii)  $\sqcup$ . Outermost parenthesis can sometimes be omitted.

We use  $\bigcap \{C_1, \dots, C_n\}$  to denote  $C_1 \sqcap \dots \sqcap C_n$ . Syntactic equality between concepts or individuals is denoted by  $=$ .

We say that  $C$  is a (*syntactic*) *subconcept* of a concept  $D$  if  $C \in \text{sc}(D)$  where  $\text{sc}$  is defined as follows.

$$\begin{aligned} \text{sc}(C) &= \{C\} \text{ if } C \in \{A, \top, \perp\} \\ \text{sc}(\neg C) &= \text{sc}(C) \cup \{\neg C\} \\ \text{sc}(C \sqcap D) &= \text{sc}(C) \cup \text{sc}(D) \cup \{C \sqcap D\} \\ \text{sc}(C \sqcup D) &= \text{sc}(C) \cup \text{sc}(D) \cup \{C \sqcup D\} \\ \text{sc}(\forall R.C) &= \text{sc}(C) \cup \{\forall R.C\} \\ \text{sc}(\exists R.C) &= \text{sc}(C) \cup \{\exists R.C\} \end{aligned}$$

A knowledge base  $\mathcal{K}$  in  $\mathcal{ALC}$  is a pair  $(\mathcal{T}, \mathcal{A})$  where

1.  $\mathcal{T}$ , called a *Tbox*, is a finite set of axioms of the form  $C \sqsubseteq D$ , with  $C, D$  any two concepts. Statements of the form  $C \equiv D$  are abbreviations for  $C \sqsubseteq D$  and  $D \sqsubseteq C$ .
2.  $\mathcal{A}$ , called an *Abox*, is a finite set of statements of the form  $C(a)$ ,  $R(a, b)$ ,  $a = b$  or  $a \neq b$ .

The set of all individuals occurring in  $\mathcal{A}$  is denoted by  $\text{dom}(\mathcal{A})$ .

To avoid confusion with the syntactic equality, for the statements of the Abox we always write the information of it, i.e.  $a = b \in \mathcal{A}$ .

Note that Aboxes contain equalities and inequalities between individuals in spite of the fact that they are not part of the standard definition of  $\mathcal{ALC}$ . There are two reasons for adding them. First of all, this is a very useful OWL feature. Second and most important, it makes it evident that equality and difference between individuals play an important role in the presence of meta-modelling since an equality between individuals is transferred into an equality between the corresponding concepts and conversely.

An *interpretation*  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$  consists of a non-empty set  $\Delta^{\mathcal{I}}$  (sometimes we drop the super-index when the name of the interpretation is clear from the context and write just  $\Delta$ ), called the *domain* of  $\mathcal{I}$ , and a function  $\cdot^{\mathcal{I}}$  which maps every concept to a subset of  $\Delta$  and every role to a subset of  $\Delta \times \Delta$  such that, for all concepts  $C, D$  and role  $R$  the following equations are satisfied:

$$\begin{aligned} \top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\ (C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\ (C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\ (\neg C)^{\mathcal{I}} &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\ (\exists R.C)^{\mathcal{I}} &= \{x \mid \exists y. (x, y) \in R^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\} \\ (\forall R.C)^{\mathcal{I}} &= \{x \mid \forall y. (x, y) \in R^{\mathcal{I}} \text{ implies } y \in C^{\mathcal{I}}\} \end{aligned}$$

An interpretation  $\mathcal{I}$  *satisfies a concept*  $C$ , denoted by  $\mathcal{I} \models C$ , if  $C^{\mathcal{I}} \neq \emptyset$  and it satisfies a set  $\mathcal{X}$  of concepts, denoted by  $\mathcal{I} \models \mathcal{X}$ , if  $(\bigcap \mathcal{X})^{\mathcal{I}} \neq \emptyset$ . Note that  $(\bigcap \mathcal{X})^{\mathcal{I}} = \bigcap_{C \in \mathcal{X}} C^{\mathcal{I}}$ . An interpretation  $\mathcal{I}$  *satisfies a TBox*  $\mathcal{T}$ , denoted by  $\mathcal{I} \models \mathcal{T}$ , if  $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$  for each  $C \sqsubseteq D$  in  $\mathcal{T}$ .

An interpretation  $\mathcal{I}$  *satisfies a set*  $\mathcal{X}$  *of concepts w.r.t. Tbox*  $\mathcal{T}$  or  $\mathcal{I}$  *satisfies*  $(\mathcal{T}, \mathcal{X})$ , denoted by  $\mathcal{I} \models (\mathcal{T}, \mathcal{X})$ , if  $\mathcal{I}$  satisfies  $\mathcal{T}$  and  $\mathcal{X}$ .

An interpretation  $\mathcal{I}$  *validates a concept*  $C$ , denoted as  $\mathcal{I} \models C \equiv \top$ , if  $C^{\mathcal{I}} = \Delta^{\mathcal{I}}$ .

An interpretation  $\mathcal{I}$  *validates a set*  $\mathcal{X}$  *of concepts* if  $\mathcal{I}$  validates every concept in  $\mathcal{X}$ , or equivalently  $\mathcal{I} \models \bigcap \mathcal{X} \equiv \top$ .

An interpretation  $\mathcal{I}$  *satisfies an ABox*  $\mathcal{A}$ , denoted by  $\mathcal{I} \models \mathcal{A}$ , if  $a^{\mathcal{I}} \in C^{\mathcal{I}}$  for each  $C(a)$  in  $\mathcal{A}$ ,  $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$  for each  $R(a, b)$  in  $\mathcal{A}$ ,  $a^{\mathcal{I}} = b^{\mathcal{I}}$  for each  $a = b$  in  $\mathcal{A}$  and  $a^{\mathcal{I}} \neq b^{\mathcal{I}}$  for each  $a \neq b$  in  $\mathcal{A}$ .

An interpretation  $\mathcal{I}$  is a *model* of  $(\mathcal{T}, \mathcal{A})$ , denoted by  $\mathcal{I} \models (\mathcal{T}, \mathcal{A})$  if it satisfies the Tbox  $\mathcal{T}$  and the Abox  $\mathcal{A}$ .

We say that a knowledge base  $\mathcal{K} = (\mathcal{T}, \mathcal{A})$  is *consistent* (or *satisfiable*) if there exists a model of  $\mathcal{K}$ .

We say that a concept is in *negation normal form* if negation occurs in front of atomic

concepts only. The negation normal form of a concept is denoted by  $\text{NNF}(C)$  and defined in Figure 2. An Abox and a Tbox are also converted into negation normal form. An axiom  $C \equiv D$  in  $\mathcal{T}$  is converted into  $\text{NNF}(\neg C \sqcup D) \sqcap \text{NNF}(\neg D \sqcup C)$ . Actually, a concept  $C$  in the set  $\text{NNF}(\mathcal{T})$  which is in negation normal form represents the axiom  $C \equiv \top$ . This means that  $x \in C^{\mathcal{I}}$  for all  $x$  in the domain  $\Delta^{\mathcal{I}}$  and all  $C$  in  $\text{NNF}(\mathcal{T})$ . Thus, an interpretation  $\mathcal{I}$  is a model of  $\mathcal{T}$  if and only if  $\mathcal{I}$  validates every concept  $C \in \text{NNF}(\mathcal{T})$ . When  $\mathcal{T}$  is in negation normal form, then  $\mathcal{T}$  is not a set of inclusions but just a set of concepts  $C$  such that  $C \equiv \top$  should hold. In that case, we say that  $\mathcal{I}$  is a model of  $\mathcal{T}$ , denoted by  $\mathcal{I} \models \sqcap \mathcal{T} \equiv \top$ , if  $\mathcal{I}$  validates every concept  $C \in \mathcal{T}$ .

$$\begin{aligned}
\text{NNF}(A) &= A \quad \text{if } A \text{ is an atomic concept} \\
\text{NNF}(\neg A) &= \neg A \quad \text{if } A \text{ is an atomic concept} \\
\text{NNF}(\neg\neg C) &= \text{NNF}(C) \\
\text{NNF}(C \sqcup D) &= \text{NNF}(C) \sqcup \text{NNF}(D) \\
\text{NNF}(C \sqcap D) &= \text{NNF}(C) \sqcap \text{NNF}(D) \\
\text{NNF}(\neg(C \sqcup D)) &= \text{NNF}(\neg C) \sqcap \text{NNF}(\neg D) \\
\text{NNF}(\neg(C \sqcap D)) &= \text{NNF}(\neg C) \sqcup \text{NNF}(\neg D) \\
\text{NNF}(\forall R.C) &= \forall R.\text{NNF}(C) \\
\text{NNF}(\exists R.C) &= \exists R.\text{NNF}(C) \\
\text{NNF}(\neg\forall R.C) &= \exists R.\text{NNF}(\neg C) \\
\text{NNF}(\neg\exists R.C) &= \forall R.\text{NNF}(\neg C) \\
\\
\text{NNF}(\mathcal{T}) &= \bigcup_{C \sqsubseteq D \in \mathcal{T}} \text{NNF}(\neg C \sqcup D) \\
\text{NNF}(\mathcal{A}) &= \bigcup_{C(a) \in \mathcal{A}} \text{NNF}(C)(a) \cup \bigcup_{R(a,b) \in \mathcal{A}} R(a,b) \cup \\
&\quad \bigcup_{a=b \in \mathcal{A}} a = b \cup \bigcup_{a \neq b \in \mathcal{A}} a \neq b
\end{aligned}$$

**Fig. 2.** Negation Normal Form of a Concept, a TBox and an ABox

**Definition 1 (Isomorphism between interpretations of  $\mathcal{ALC}$ ).**

An isomorphism between two interpretations  $\mathcal{I}$  and  $\mathcal{I}'$  of  $\mathcal{ALC}$  is a bijective function  $f : \Delta \rightarrow \Delta'$  such that

- $f(a^{\mathcal{I}}) = a^{\mathcal{I}'}$
- $x \in A^{\mathcal{I}}$  if and only if  $f(x) \in A^{\mathcal{I}'}$
- $(x, y) \in R^{\mathcal{I}}$  if and only if  $(f(x), f(y)) \in R^{\mathcal{I}'}$ .

**Lemma 1.** Let  $\mathcal{I}$  and  $\mathcal{I}'$  be two isomorphic interpretations of  $\mathcal{ALC}$ . Then,  $\mathcal{I}$  is a model of  $(\mathcal{T}, \mathcal{A})$  if and only if  $\mathcal{I}'$  is a model of  $(\mathcal{T}, \mathcal{A})$ .

To prove the previous lemma is enough to show that  $x \in C^{\mathcal{I}}$  if and only if  $f(x) \in C^{\mathcal{I}'}$  by induction on  $C$ .

**Theorem 1 (Complexity of  $\mathcal{ALC}$ ).** Consistency of a (general) knowledge base in  $\mathcal{ALC}$  is *ExpTime-complete*.

The hardness result was proved by Schild [3]. A matching upper bound for  $\mathcal{ALC}$  was given by De Giacomo and Lenzerini by a reduction to PDL [4]. The classic Tableau algorithm is not optimal and cannot be used in this proof because it is NExpTime [5]. ExpTime Tableau algorithms for checking satisfiability w.r.t. a general Tbox are shown by Lenzerini, Donini and Masacci [18,19]. These algorithms globally cache only unsatisfiable sets. Goré and Nguyen show an ExpTime Tableau algorithm for checking satisfiability of a concept in  $\mathcal{ALC}$  w.r.t. a general Tbox that can globally cache satisfiable and unsatisfiable sets [20]. Nguyen and Szalas extend this same algorithm for checking consistency of a knowledge base (including a Tbox and an Abox) in  $\mathcal{ALC}$  [15].

## 4 Well-founded Sets and Well-founded Relations

In this section we recall some basic notions on well-founded sets and relations [21]. In particular, the induction and recursion principles are important for us since we will use them in the proof of completeness of the Tableau Calculus for  $\mathcal{ALCM}$ .

**Definition 2 (Well-founded Relation).** Let  $X$  be a set and  $\prec$  a binary relation on  $X$ .

1. Let  $Y \subseteq X$ . We say that  $m \in Y$  is a minimal element of  $Y$  if there is no  $y \in Y$  such that  $y \prec m$ .
2. We say that  $\prec$  is well-founded (on  $X$ ) if for all  $Y \neq \emptyset$  such that  $Y \subseteq X$ , we have that  $Y$  has a minimal element.

Note that in the general definition above the relation  $\prec$  does not need to be transitive.

**Lemma 2.** The order  $\prec$  is well-founded on  $X$  iff there are no infinite  $\prec$ -decreasing sequences, i.e., there is no  $\langle x_i \rangle_{i \in \mathbb{N}}$  such that  $x_{i+1} \prec x_i$  and  $x_i \in X$  for all  $i \in \mathbb{N}$ .

The proof of the above lemma can be found in [21].

**Definition 3 (Well-founded Set).** A set  $X$  is well-founded if the set membership relation  $\in$  is well-founded on the set  $X$ .

As a consequence of Lemma 2, we also have that:

1. If  $X$  is a well-founded set then  $X \notin X$ .
2. If  $X$  is a well-founded set then it cannot contain an infinite  $\in$ -decreasing sequence, i.e., there is no  $\langle x_n \rangle_{n \in \mathbb{N}}$  such that  $x_{n+1} \in x_n$  and  $x_n \in X$  for all  $n \in \mathbb{N}$ .

An important reason that well-founded relations are interesting is because we can apply the induction and recursion principles, e.g., [21]. In this paper both principles will be used to prove correctness of the Tableau calculus for  $\mathcal{ALCM}$ .

**Definition 4 (Induction Principle).** If  $\prec$  is a well-founded relation on  $X$ ,  $\varphi$  is some property of elements of  $X$ , and we want to show that  $\varphi(x)$  holds for all elements  $x \in X$ , it suffices to show that:

if  $x \in X$  and  $\varphi(y)$  is true for all  $y \in X$  such that  $y \prec x$ , then  $\varphi(x)$  must also be true.

**Definition 5 (Function Restriction).** The restriction of a function  $f : X \rightarrow Y$  to a subset  $X'$  of  $X$  is denoted as  $f|_{X'}$ , and defined as follows.

$$f|_{X'} = \{(x, f(x)) \mid x \in X'\}$$

On par with induction, well-founded relations also support construction of objects by recursion.

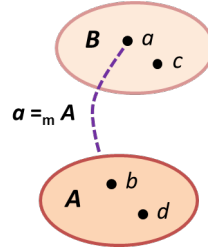
**Definition 6 (Recursion Principle).** If  $\prec$  is a well-founded relation on  $X$  and  $F$  a function that assigns an object  $F(x, g)$  to each pair of an element  $x \in X$  and a function  $g$  on the initial segment  $\{y \in X \mid y \prec x\}$  of  $X$ . Then there is a unique function  $G$  such that for every  $x \in X$ ,

$$G(x) = F(x, G|_{\{y \in X \mid y \prec x\}})$$

## 5 The Description Logic $\mathcal{ALCM}$

In this section, we extend the description logic  $\mathcal{ALC}$  with the meta-modelling defined by Motz, Rohrer and Severi [1,2].

**Definition 7 (Meta-modelling axiom).** A meta-modelling axiom is a statement of the form  $a =_m A$  where  $a$  is an individual and  $A$  is an atomic concept. We pronounce  $a =_m A$  as  $a$  corresponds to  $A$  through meta-modelling. An Mbox  $\mathcal{M}$  is a finite set of meta-modelling axioms.



**Fig. 3.** Meta-modelling Axiom Example

In Figure 3, the meta-modelling axiom  $a =_m A$  express that the individual  $a$  corresponds to the concept  $A$  through meta-modelling.

We define  $\mathcal{ALCM}$  by keeping the same syntax for concept expressions as for  $\mathcal{ALC}$ .

A knowledge base  $\mathcal{K}$  in  $\mathcal{ALCM}$  is a triple  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  where  $\mathcal{T}$ ,  $\mathcal{A}$  and  $\mathcal{M}$  are a Tbox, Abox and an Mbox respectively. The set of all individuals in  $\mathcal{M}$  is denoted by  $\text{dom}(\mathcal{M})$  and the set of all concepts by  $\text{range}(\mathcal{M})$ .

Figure 4 shows the Tbox, Abox and Mbox of the knowledge base that corresponds to Figure 1.

<b>Tbox</b>	<b>Abox</b>
$River \sqcap Lake \sqsubseteq \perp$	$HydrographicObject(river)$
	$HydrographicObject(lake)$
<b>Mbox</b>	$River(queguay)$
$river =_m River$	$River(santaLucia)$
$lake =_m Lake$	$Lake(deRocha)$
	$Lake(delSauce)$

**Fig. 4.** Tbox, Abox and Mbox for Figure 1

**Definition 8 (Satisfiability of meta-modelling).** *An interpretation  $\mathcal{I}$  satisfies (or it is a model of)  $a =_m A$  if  $a^{\mathcal{I}} = A^{\mathcal{I}}$ . An interpretation  $\mathcal{I}$  satisfies (or it is model of)  $\mathcal{M}$ , denoted by  $\mathcal{I} \models \mathcal{M}$ , if it satisfies each statement in  $\mathcal{M}$ .*

The semantics of  $\mathcal{ALCM}$  makes use of the structured domain elements. In order to give semantics to meta-modelling, the domain has to consists of basic objects, sets of objects, sets of sets of objects and so on.

**Definition 9 ( $S_n$  for  $n \in \mathbb{N}$ ).** *Given a non empty set  $S_0$  of atomic objects, we define  $S_n$  by induction on  $\mathbb{N}$  as follows:  $S_{n+1} = S_n \cup \mathcal{P}(S_n)$*

It is easy to prove that  $S_n \subseteq S_{n+1}$  and that  $S_n$  is well-founded for all  $n \in \mathbb{N}$ .

A set  $X \subseteq S_n$  can contain elements  $x$  such that  $x \in S_i$  for any  $i \leq n$ .

**Definition 10 (Model of a Knowledge Base in  $\mathcal{ALCM}$ ).** *An interpretation  $\mathcal{I}$  is a model of a knowledge base  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  in  $\mathcal{ALCM}$  (denoted as  $\mathcal{I} \models \mathcal{K}$ ) if the following holds:*

1. *the domain  $\Delta$  of the interpretation is a subset of some  $S_n$  for some  $n \in \mathbb{N}$ .*
2.  *$\mathcal{I}$  is a model of  $(\mathcal{T}, \mathcal{A})$  in  $\mathcal{ALC}$ .*
3.  *$\mathcal{I}$  is a model of  $\mathcal{M}$ .*

In the first part of Definition 10 we restrict the domain of an interpretation in  $\mathcal{ALCM}$  to be a *subset* of  $S_n$ . The domain  $\Delta$  can now contain sets since the set  $S_n$  is defined recursively using the powerset operation. In the presence of meta-modelling, the domain  $\Delta$  cannot longer consist of only basic objects and cannot be an arbitrary set either. We require that the domain be a well-founded set. The reason for this is explained as follows. Suppose we have a domain  $\Delta^{\mathcal{I}} = \{X\}$  where  $X = \{X\}$  is a set that belongs to itself. Intuitively,  $X$  is the set

$$\{\{\{\dots\}\}\}$$

Clearly, a set like  $X$  should be excluded from our interpretation domain since it cannot represent any real object from our usual applications in Semantic Web (in other areas or aspects of Computer Science, representing such objects is useful [22]).

Note that  $S_0$  does not have to be the same for all models of a knowledge base.

The second part of Definition 10 refers to the  $\mathcal{ALC}$ -knowledge base without the Mbox axioms. In the third part of the definition, we add another condition that the model must



satisfy considering the meta-modelling axioms. This condition restricts the interpretation of an individual that has a corresponding concept through meta-modelling to be equal to the concept interpretation.

*Example 3.* We define a model for the knowledge base of Figure 4 where

$$\begin{aligned} S_0 &= \{queguay, santaLucia, deRocha, delSauce\} \\ \Delta &= \{ \text{queguay, santaLucia, deRocha, delSauce,} \\ &\quad \{queguay, santaLucia\}, \\ &\quad \{deRocha, delSauce\}, \\ &\quad \{\{queguay, santaLucia\}, \{deRocha, delSauce\}\} \\ &\quad \} \end{aligned}$$

The interpretation is defined on the individuals with meta-modelling and the corresponding atomic concepts to which they are equated as follows:

$$\begin{aligned} river^{\mathcal{I}} &= River^{\mathcal{I}} = \{queguay, santaLucia\} \\ lake^{\mathcal{I}} &= Lake^{\mathcal{I}} = \{deRocha, delSauce\} \end{aligned}$$

and on the remaining atomic concept which does not appear on the MBox the interpretation is defined as follows:

$$\begin{aligned} HydrographicObject^{\mathcal{I}} & \\ &= \{river^{\mathcal{I}}, lake^{\mathcal{I}}\} \\ &= \{\{queguay, santaLucia\}, \{deRocha, delSauce\}\} \end{aligned}$$

**Definition 11 (Consistency in  $\mathcal{ALCM}$ ).** We say that a knowledge base  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  is consistent (satisfiable) if there exists a model of  $\mathcal{K}$ .

**Definition 12 (Logical Consequence in  $\mathcal{ALCM}$ ).** We say that  $S$  is a logical consequence of  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  (denoted as  $\mathcal{K} \models S$ ) if all models of  $\mathcal{K}$  are also models of  $S$  where  $S$  is any of the following  $\mathcal{ALCM}$  statements, i.e.,

$$C \sqsubseteq D \quad C(a) \quad R(a, b) \quad a =_m A \quad a = b \quad a \neq b.$$

**Definition 13 (Meta-concept).** We say that  $C$  is a meta-concept in  $\mathcal{K}$  if there exists an individual  $a$  such that  $\mathcal{K} \models C(a)$  and  $\mathcal{K} \models a =_m A$ .

Then,  $C$  is a meta-meta-concept if there exists an individual  $a$  such that  $\mathcal{K} \models C(a)$ ,  $\mathcal{K} \models a =_m A$  and  $A$  is a meta-concept. Note that a meta-meta-concept is also a meta-concept.

We have some new inference problems:

1. *Meta-modelling.* Find out whether  $a =_m A$  or not.
2. *Meta-concept.* Find out whether  $C$  is a meta-concept or not.

As most inference problems in Description Logic the above two problems can be reduced to satisfiability. For the first problem, note that since  $a \neq_m A$  is not directly available in the syntax, we have replaced it by  $a \neq b$  and  $b =_m A$  which is an equivalent statement to the negation of  $a =_m A$ . For the proof of the following two lemmas, see [1].

**Lemma 3.**  $\mathcal{K} \models a =_{\text{m}} A$  if and only if for some new individual  $b$ ,  $\mathcal{K} \cup \{a \neq b, b =_{\text{m}} A\}$  is inconsistent.

**Lemma 4.** A concept  $C$  is a meta-concept if and only if for some individual  $a$  we have that  $\mathcal{K} \cup \{\neg C(a)\}$  is inconsistent and for some new individual  $b$ ,  $\mathcal{K} \cup \{a \neq b, b =_{\text{m}} A\}$  is inconsistent.

Next lemma explains more formally why in our definition of ABox we included expressions of the form  $a = b$  and  $a \neq b$ . If we have an equality  $A \equiv B$  between concepts then  $a$  and  $b$  should be equal. Similarly, if we have that  $A$  and  $B$  are different then  $a$  and  $b$  should be different. In other words, since we can express equality and difference between concepts, we also need to be able to express equality and difference at the level of individuals.

**Lemma 5 (Equality Transference).**

Let  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  be a knowledge base,  $\mathcal{K} \models a =_{\text{m}} A$  and  $\mathcal{K} \models b =_{\text{m}} B$ .

1. If  $\mathcal{K} \models a = b$  then  $\mathcal{K} \models A \equiv B$ .
2. If  $\mathcal{K} \models A \equiv B$  then  $\mathcal{K} \models a = b$ .

The proof of the above lemma is immediate since  $a$ ,  $b$ ,  $A$  and  $B$  are all interpreted as the same object.

We define a tableau algorithm for checking consistency of a knowledge base in  $\mathcal{ALCM}$  by extending the standard tableau algorithm for  $\mathcal{ALC}$ . The expansion rules for  $\mathcal{ALCM}$  consist of the rules for  $\mathcal{ALC}$  and some additional expansion rules for meta-modelling (see Figure 5). The additional expansion rules deal with the equalities and inequalities between individuals with meta-modelling which need to be transferred to the level of concepts as equalities and inequalities between the corresponding concepts. We also need to add an extra condition that checks for circularities (with respect to membership) avoiding non well-founded sets.

A completion forest  $\mathcal{L}$  for an  $\mathcal{ALC}$  knowledge base consists of

1. a set of nodes, labelled with individual names or variable names (fresh individuals which do not belong to the ABox),
2. directed edges between some pairs of nodes,
3. for each node labelled  $x$ , a set  $\mathcal{L}(x)$  of concept expressions,
4. for each pair of nodes  $x$  and  $y$ , a set  $\mathcal{L}(x, y)$  containing role names, and
5. two relations between nodes, denoted by  $\approx$  and  $\not\approx$ . These relations keep record of the equalities and inequalities of nodes in the algorithm. The relation  $\approx$  is assumed to be reflexive, symmetric and transitive while  $\not\approx$  is assumed to be symmetric. We also assume that the relation  $\not\approx$  is compatible with  $\approx$ , i.e., if  $x' \approx x$  and  $x \not\approx y$  then  $x' \not\approx y$  for all  $x, x', y$ . In the algorithm, every time we add a pair in  $\approx$ , we close  $\approx$  under reflexivity, symmetry and transitivity. Moreover, every time we add a pair in either  $\not\approx$  or  $\approx$ , we close  $\not\approx$  under compatibility with  $\approx$ .

We assume that  $\mathcal{T}$  y  $\mathcal{A}$  have already been converted into negation normal form.

**Definition 14 (Initialization).** The initial completion forest for  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  is defined by the following procedure.

1. For each individual  $a$  in the knowledge ( $a \in \mathcal{A} \cup \mathcal{M}$ ) set  $a \approx a$ .
2. For each  $a = b \in \mathcal{A}$ , set  $a \approx b$ . We also choose an individual as a representative of each equivalence class.
3. For each  $a \neq b$  in  $\mathcal{A}$ , set  $a \not\approx b$ .
4. For each  $a \in \mathcal{A} \cup \mathcal{M}$ , we do the following:
  - (a) in case  $a$  is a representative of an equivalence class then set  $\mathcal{L}(a) = \{C \mid C(a') \in \mathcal{A}, a \approx a'\}$ ;
  - (b) in case  $a$  is not a representative of an equivalence class then set  $\mathcal{L}(a) = \emptyset$ .
5. For all  $a, b \in \mathcal{A} \cup \mathcal{M}$  that are representatives of some equivalence class, if  $\{R \mid R(a', b') \in \mathcal{A}, a \approx a', b \approx b'\} \neq \emptyset$  then create an edge from  $a$  to  $b$  and set  $\mathcal{L}(a, b) = \{R \mid R(a', b') \in \mathcal{A}, a \approx a', b \approx b'\}$ .

Note that in case  $a$  is not a representative of an equivalence class and it has some axiom  $C(a)$ , we set  $\mathcal{L}(a) = \emptyset$  because we do not want to apply any expansion rule to  $\mathcal{L}(a)$ . The expansion rules will only be applied to the representative of the equivalence class of  $a$ .

**Definition 15 (Contradiction).** A completion forest  $\mathcal{L}$  has a contradiction if either

- $A$  and  $\neg A$  belongs to  $\mathcal{L}(x)$  for some atomic concept  $A$  and node  $x$  or
- there are nodes  $x$  and  $y$  such that  $x \not\approx y$  and  $x \approx y$ .

We say that a node  $y$  is a *successor* of a node  $x$  if  $\mathcal{L}(x, y) \neq \emptyset$ . We define that  $y$  is a *descendant* of  $x$  by induction.

1. Every successor of  $x$ , which is a variable, is a descendant of  $x$ .
2. Every successor of a descendant of  $x$ , which is a variable, is also a descendant of  $x$ .

**Definition 16 (Blocking).** We define the notion of blocking by induction. A node  $x$  is blocked by a node  $y$  if  $x$  is a descendant of  $y$  and  $\mathcal{L}(x) \subseteq \mathcal{L}(y)$  or  $x$  is a descendant of  $z$  and  $z$  is blocked by  $y$ .

**Definition 17 (ALCM-Complete).** A completion forest  $\mathcal{L}$  is *ALCM-complete* (or just complete) if none of the rules of Figure 5 is applicable.

**Definition 18 (Circularity).** We say that the completion forest  $\mathcal{L}$  has a circularity with respect to  $\mathcal{M}$  if there is a sequence of meta-modelling axioms  $a_0 =_{\mathcal{M}} A_0, a_1 =_{\mathcal{M}} A_1, \dots, a_n =_{\mathcal{M}} A_n$  all in  $\mathcal{M}$  such that

$$\begin{array}{ll}
 A_1 \in \mathcal{L}(x_0) & x_0 \approx a_0 \\
 A_2 \in \mathcal{L}(x_1) & x_1 \approx a_1 \\
 \vdots & \vdots \\
 A_n \in \mathcal{L}(x_{n-1}) & x_{n-1} \approx a_{n-1} \\
 A_0 \in \mathcal{L}(x_n) & x_n \approx a_n
 \end{array}$$

After initialization, the tableau algorithm proceeds by non-deterministically applying the *expansion rules for  $\mathcal{ALCCM}$*  given in Figure 5.

The algorithm says that the ontology  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  is consistent iff the expansion rules can be applied in such a way they yield a complete forest  $\mathcal{L}$  without contradictions nor circularities. Otherwise the algorithm says that it is inconsistent. Note that due to the non-determinism of the algorithm, implementations of it have to guess the choices and possibly have to backtrack to choice points if a choice already made has led to a contradiction. The algorithm stops when we reach *some*  $\mathcal{L}$  that is complete, has neither contradictions nor circularities or when all the choices have yield a forest with contradictions or circularities.

**$\sqcap$ -rule:**

If  $C \sqcap D \in \mathcal{L}(x)$  and  $\{C, D\} \not\subseteq \mathcal{L}(x)$  then set  $\mathcal{L}(x) \leftarrow \{C, D\}$ .

**$\sqcup$ -rule:**

If  $C \sqcup D \in \mathcal{L}(x)$  and  $\{C, D\} \cap \mathcal{L}(x) = \emptyset$  then set  $\mathcal{L}(x) \leftarrow \{C\}$  or  $\mathcal{L}(x) \leftarrow \{D\}$ .

**$\exists$ -rule:**

If  $x$  is not blocked,  $\exists R.C \in \mathcal{L}(x)$  and there is no  $y$  with  $R \in \mathcal{L}(x, y)$  and  $C \in \mathcal{L}(y)$  then

1. Add a new node with label  $y$  (where  $y$  is a new node label), 2. set  $\mathcal{L}(x, y) = \{R\}$ ,

3. set  $\mathcal{L}(y) = \{C\}$ .

**$\forall$ -rule:**

If  $\forall R.C \in \mathcal{L}(x)$  and there is a node  $y$  with  $R \in \mathcal{L}(x, y)$  and  $C \notin \mathcal{L}(y)$  then set  $\mathcal{L}(x) \leftarrow C$ .

**$\mathcal{T}$ -rule:**

If  $C \in \mathcal{T}$  and  $C \notin \mathcal{L}(x)$ , then  $\mathcal{L}(x) \leftarrow C$ .

**$\approx$ -rule:**

Let  $a =_m A$  and  $b =_m B$  in  $\mathcal{M}$ . If  $a \approx b$  and  $A \sqcup \neg B, B \sqcup \neg A$  does not belong to  $\mathcal{T}$  then add  $A \sqcup \neg B, B \sqcup \neg A$  to  $\mathcal{T}$ .

**$\not\approx$ -rule:**

Let  $a =_m A$  and  $b =_m B$  in  $\mathcal{M}$ . If  $a \not\approx b$  and there is no node  $z$  such that  $A \sqcap \neg B \sqcup B \sqcap \neg A \in \mathcal{L}(z)$  then create a new node  $z$  with  $\mathcal{L}(z) = \{A \sqcap \neg B \sqcup B \sqcap \neg A\}$

**close-rule:**

Let  $a =_m A$  and  $b =_m B$  in  $\mathcal{M}$  where  $a \approx x, b \approx y, x$  and  $y$  are their respective representatives of the equivalence classes. If neither  $x \approx y$  nor  $x \not\approx y$  then we add either  $x \approx y$  or  $x \not\approx y$ . In the case  $x \approx y$ , we also do the following:

1. add  $\mathcal{L}(y)$  to  $\mathcal{L}(x)$ ,
2. for all directed edges from  $y$  to some  $w$ , create an edge from  $x$  to  $w$  if it does not exist with  $\mathcal{L}(x, w) = \emptyset$ ,
3. add  $\mathcal{L}(y, w)$  to  $\mathcal{L}(x, w)$ ,
4. for all directed edges from some  $w$  to  $y$ , create an edge from  $w$  to  $x$  if it does not exist with  $\mathcal{L}(w, x) = \emptyset$ ,
5. add  $\mathcal{L}(w, y)$  to  $\mathcal{L}(w, x)$ ,
6. set  $\mathcal{L}(y) = \emptyset$  and remove all edges from/to  $y$ .

**Fig. 5.** Expansion Rules for  $\mathcal{ALCCM}$

This tableau algorithm has complexity NEXP and hence, it is not optimal. In the following section we define an optimal algorithm for  $\mathcal{ALCCM}$  and prove that is ExpTime.

## 6 A Tableau Calculus for $\mathcal{ALCM}$

In order to eliminate non-determinism, the completion trees of the standard tableau algorithm are replaced with structures called *and-or graphs* [15]. In such structure, both branches of a non-deterministic choice introduced by disjunction are explicitly represented. Satisfiability of the branches is propagated bottom-up and if it reaches an initial node, we can be sure that a model exists. A global catching of nodes and a proper rule-application strategy is used to guarantee the exponential bound on the size of the graph. The and-or graphs are built using a Tableau Calculus. Our Tableau Calculus for  $\mathcal{ALCM}$  is an extension of the Tableau Calculus for  $\mathcal{ALC}$  given by Nguyen and Szalas [15], which adds new rules for handling meta-modelling.

From now on, we assume that Tboxes, Aboxes and all concepts are in negation normal form (NNF) (see Figure 2). Note that the concepts in an Mbox are already in NNF since they are all atomic.

The Tableau Calculus for  $\mathcal{ALCM}$  is defined by the tableau rules of Figures 6 and 7. In the premises and conclusions of these rules, we find *judgements*  $J$  that are built using the following grammar.

1. *Simple judgements*  $J$  are of the following three forms:

$$(\mathcal{T}, \mathcal{A}, \mathcal{M}) \quad \text{base judgement}$$

$$(\mathcal{T}, \mathcal{X}) \quad \text{variable judgement}$$

$$\perp \quad \text{absurdity judgement}$$

2. *Or-judgements* are of the form  $J_1 \vee J_2$ , where  $J_1$  and  $J_2$  are simple judgements.
3. *And-judgements* are of the form  $J_1 \wedge \dots \wedge J_k$ , where  $J_i$  are simple judgements for all  $1 \leq i \leq k$ .

The first basic judgement is just the knowledge base  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ . After initialization, the Aboxes  $\mathcal{A}$  of the Tableau Calculus do not contain equality assertions  $a = b$  because we choose the individual  $a$  as a canonical representative and replace  $b$  by  $a$ . We write  $\mathcal{A}[a/b]$  and  $\mathcal{M}[a/b]$  to denote the replacement of  $b$  by  $a$  in  $\mathcal{A}$  and  $\mathcal{M}$  respectively.

For the second basic judgement, we have that  $\mathcal{T}$  is a Tbox and  $\mathcal{X}$  is a set of concepts. The set  $\mathcal{X}$  intuitively represents the set of concepts satisfiable by a certain unknown  $x$ . The key feature of this tableau calculus is actually not to create variables  $x$  for the existential restrictions as the standard tableau for Description Logic does. It actually forgets the variables and only cares about the set  $\mathcal{X}$  of concepts that should be satisfiable. By not writing  $x$  explicitly, the set  $\mathcal{X}$  can be shared by another existential coming from another branch in the graph and this will allow us to obtain an ExpTime algorithm for checking consistency.

Recall from Definition 11 and 10 that  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  is satisfiable if there exists  $\mathcal{I}$  such that

1.  $\mathcal{I} \models \bigcap \mathcal{T} \equiv \top$  thus  $\mathcal{I}$  is a model of  $\mathcal{T}$ ,
2.  $\mathcal{I} \models \mathcal{A}$  and
3.  $\mathcal{I} \models \mathcal{M}$ .

Recall that  $(\mathcal{T}, \mathcal{X})$  is satisfiable if there exists  $\mathcal{I}$  such that

1.  $\mathcal{I} \models \bigcap \mathcal{T} \equiv \top$  thus  $\mathcal{I}$  is a model of  $\mathcal{T}$  and
2.  $\mathcal{I}$  satisfies the set  $\mathcal{X}$  of concepts, i.e.  $(\bigcap \mathcal{X})^{\mathcal{I}} \neq \emptyset$ .

We have that  $\perp$  is always unsatisfiable.

We say that  $J_1 \wedge \dots \wedge J_k$  is satisfiable if  $J_i$  is satisfiable for all  $1 \leq i \leq k$ . We use  $\bigwedge \{J_1, \dots, J_k\}$  to denote  $J_1 \wedge \dots \wedge J_k$ .

We say that  $J_1 \vee J_2$  is satisfiable if  $J_1$  or  $J_2$  is satisfiable.

The tableau rules are written downwards, they have only one simple judgement as premise and can have a sequence of simple judgements as conclusion. *Unary rules* have only a simple judgement as conclusion and they are of the form:

$$\frac{J_0}{J_1}$$

where  $J_0$  and  $J_1$  are both simple.

*Non-unary rules* have a conclusion which is a and-judgements or an or-judgements. In the first case, we have an *and*-rule which is of the form:

$$\frac{J_0}{J_1 \wedge \dots \wedge J_k}$$

where  $J_i$  is simple for all  $1 \leq i \leq k$ . In this case, the arity of the rule is  $k$ .

When the conclusion is an or-judgements, we have an *or*-rule which is of the form:

$$\frac{J_0}{J_1 \vee J_2}$$

where  $J_1$  and  $J_2$  are simple. In this case, the arity of the rule is 2.

A *bottom rule* is a rule whose conclusion is  $\perp$ . The (trans) and (trans') rules are called *transitional*, the rest of the rules are called *static*.

The rules  $(\sqcup)$ ,  $(\sqcup')$  and (close) are all or-rules and all of them have arity 2. The rules (trans) and (trans') are and-rules. The arity of the and-rules depend on the number of existentials in  $\mathcal{A}$  or in  $\mathcal{X}$ . The rest of the rules are unary since their conclusion is a simple judgement.

In the Tableau Calculus, the rules  $(\sqcup)$ ,  $(\sqcup')$  and (close) are deterministic. The inherent choice of these rules is explicitly represented using the or-judgement.

The Tableau Calculus does not have an explicit Tbox-rule as the standard tableau algorithm. Instead, this rule is “spread inside other rules”, namely (trans), (trans') and  $(\neq)$ -rules ensuring that the new individuals satisfy the concepts in the Tbox. Also, the initialization ensures that all the individuals of the initial knowledge base satisfy the concepts of the Tbox (see Definition 21).

The rules (trans) and (trans') need to be and-rules because all the outermost existentials are treated simultaneously<sup>3</sup>. Each of these rules is a compact way of doing what was necessary to do with three rules,  $\exists$ -rule,  $\forall$ -rule and the Tbox-rule in the standard tableau algorithm.

<sup>3</sup> The concept  $\exists R.\exists S.C$  has two nested existentials and the outermost one is  $\exists R$ .

The rules (trans) and (trans') are indirectly introducing unknown individuals for each outermost existential. Those unknown individuals should satisfy all concepts in the Tbox and inherit all concepts from the  $\forall$  of the corresponding role.

We explain the intuition behind the new rules that deal with meta-modelling, which are  $(\perp_3)$ , (close),  $(=)$  and  $(\neq)$ . If  $a =_m A$  and  $b =_m B$  then the individuals  $a$  and  $b$  represent concepts. Any equality at the level of individuals should be transferred as an equality between concepts and similarly with the difference.

Note that the Aboxes of the Tableau Calculus do not contain equality assertions  $a = b$  because we choose the individual  $a$  as a canonical representative and replace  $b$  by  $a$  in the initialization as well as in the (close)-rule.

The (close)-rule is an or-rule that distinguishes between the two possibilities: either  $a$  and  $b$  are equal or they are different. In the case  $a$  and  $b$  are equal, we choose  $a$  as canonical representative and replace  $b$  by  $a$ . Note that  $b$  is replaced by  $a$  in the Mbox and we get  $a =_m B$  as the result of replacing  $b$  by  $a$  in  $b =_m B$ . In the case  $a$  and  $b$  are different, we simply add the axiom  $a \neq b$  to the Abox.

The  $(=)$ -rule transfers an “equality between individuals” to the level of concepts. Since we choose canonical representatives for individuals, this happens only when  $a =_m A$  and  $a =_m B$ . Instead of adding the statement  $A \equiv B$  to the TBox, we add its negation normal form which is  $(A \sqcap \neg B) \sqcup (\neg A \sqcap B)$ . Since we add this new concept to the TBox, we also have to add that all the existing individuals satisfy this concept. Note that the principal premise  $a =_m B$  is removed from the Mbox and does not appear in the conclusions of the rule.

The  $(\neq)$ -rule transfers the difference between individuals to the level of concepts. If  $a \neq b$  is in the Abox, then we should add that  $A \not\equiv B$ . However, we cannot add  $A \not\equiv B$  because the negation of  $\equiv$  is not directly available in the language. So, what we do is to replace it by an equivalent statement, i.e. we add an element  $d_0$  that is in  $A$  but not in  $B$  or it is in  $B$  but it is not in  $A$ . Note also that the individual  $d_0$  should also satisfy all the concepts that are in the Tbox  $\mathcal{T}$ .

The  $(\perp_3)$ -rule ensures that there are no circularities and hence, the domain of the canonical interpretation is well-founded. This rule uses the following definition:

**Definition 19 (Circularity of an Abox w.r.t an Mbox).** We say that  $\mathcal{A}$  has a circularity w.r.t.  $\mathcal{M}$ , denoted as  $\text{circular}(\mathcal{A}, \mathcal{M})$ , if there is a sequence of meta-modelling axioms  $a_1 =_m A_1, a_2 =_m A_2, \dots, a_n =_m A_n$  all in  $\mathcal{M}$  such that  $A_1(a_2), A_2(a_3), \dots, A_n(a_1)$  are in  $\mathcal{A}$ .

For example, the Abox  $\mathcal{A} = \{A(a), B(b)\}$  has a circularity w.r.t to the Mbox  $\mathcal{M} = \{a =_m A, b =_m B\}$ .

A *formula* is either a concept, or an Abox statement or an Mbox statement. Formulas are denoted by greek letters  $\varphi$ , etc. The distinguished formulas of the premise are called the *principal formulas* of the rule.

*Remark 1.*

- Note that the rules  $(\sqcup')$ ,  $(\sqcap')$ ,  $(\forall)$ ,  $(\neq)$  and (close) need a side condition to ensure termination (otherwise the same rule can be vacuously applied infinite times). The

(=)-rule does not need a side condition because the axiom  $a =_m B$  is removed from the Mbox.

- Note that the rules  $(\sqcup')$  and  $(\sqcap')$  keep the principal formula while the rules  $(\sqcup)$  and  $(\sqcap)$  do not. For the rules  $(\sqcup')$  and  $(\sqcap')$ , we cannot remove the principal formula in the conclusion because this could lead to infinite applications of these rules when we combine any of these rules with the  $(\forall)$ -rule.

$$\begin{array}{cc}
 (\perp) \frac{(\mathcal{T}, \mathcal{X} \cup \{A, \neg A\})}{\perp} & (\sqcup) \frac{(\mathcal{T}, \mathcal{X} \cup \{C \sqcup D\})}{(\mathcal{T}, \mathcal{X} \cup \{C\}) \vee (\mathcal{T}, \mathcal{X} \cup \{D\})} \\
 (\sqcap) \frac{(\mathcal{T}, \mathcal{X} \cup \{C \sqcap D\})}{(\mathcal{T}, \mathcal{X} \cup \{C, D\})} & (\text{trans}) \frac{(\mathcal{T}, \mathcal{X})}{\bigwedge \{(\mathcal{T}, \mathcal{X}_{\exists R.C}) \mid \exists R.C \in \mathcal{X}\}} \\
 \text{where } \mathcal{X}_{\exists R.C} = \{C\} \cup \{D \mid \forall R.D \in \mathcal{X}\} \cup \mathcal{T}
 \end{array}$$

**Fig. 6.** Tableau rules for variable judgements

**Definition 20 (Preferences).** *The rules are applied in the following order:*

1. The bottom rules – which are  $(\perp)$ ,  $(\perp_1)$ ,  $(\perp_2)$  and  $(\perp_3)$  – are applied with higher priority.
2. The rest of the unary rules – which are  $(\sqcap)$ ,  $(\forall)$ ,  $(\sqcap')$ ,  $(=)$  and  $(\neq)$  – are applied only if no bottom rule is applicable.
3. The  $(\sqcup)$ ,  $(\sqcup')$  and (close)-rules are applied only if no unary rule is applicable.
4. The (trans) and (trans')-rules are applied if no other rule is applicable.

We now describe how to construct an *and-or graph* from the tableau calculus of Figures 6 and 7. This graph is built using *global caching*, i.e. the graph contains at most one node with that judgement as label and this node is processed (expanded) only once. The complexity of the tableau calculus without global caching would be double exponential. But using global caching, we do not repeat nodes and the complexity is exponential.

**Definition 21 (And-or graph for a knowledge base in  $\mathcal{ALCM}$ ).** *Let  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  be a knowledge base in  $\mathcal{ALCM}$ . The and-or graph for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ , also called tableau for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ , is a graph  $\mathbb{G}$  constructed as follows.*

1. The graph contains nodes of three kinds: variable nodes, base nodes and absurdity nodes. The label of a variable node is  $(\mathcal{T}, \mathcal{X})$ . The label of a base node is  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ . The label of an absurdity node is  $\perp$ .
2. The root of the graph is a base node whose label is  $(\mathcal{T}_0, \mathcal{A}_0, \mathcal{M}_0)$  where

$$\begin{aligned}
 \mathcal{T}_0 &:= \mathcal{T} \\
 \mathcal{A}_0 &:= \mathcal{A}^* \cup \{C(a) \mid C \in \mathcal{T}, a \in \text{dom}(\mathcal{A}^*) \cup \text{dom}(\mathcal{M}^*)\} \\
 \mathcal{M}_0 &:= \mathcal{M}^*
 \end{aligned}$$



$$\begin{array}{l}
(\perp_1) \frac{(\mathcal{T}, \mathcal{A} \cup \{B(a), \neg B(a)\}, \mathcal{M})}{\perp} \quad (\sqcap') \frac{(\mathcal{T}, \mathcal{A} \cup \{(C \sqcap D)(a)\}, \mathcal{M})}{(\mathcal{T}, \mathcal{A}', \mathcal{M})} \begin{array}{l} C(a) \notin \mathcal{A} \\ \text{and} \\ D(a) \notin \mathcal{A} \end{array} \\
\text{where } \mathcal{A}' = \mathcal{A} \cup \{(C \sqcap D)(a), C(a), D(a)\} \\
(\perp_2) \frac{(\mathcal{T}, \mathcal{A} \cup \{a \neq a\}, \mathcal{M})}{\perp} \quad (\sqcup') \frac{(\mathcal{T}, \mathcal{A} \cup \{(C \sqcup D)(a)\}, \mathcal{M})}{(\mathcal{T}, \mathcal{A}', \mathcal{M}) \vee (\mathcal{T}, \mathcal{A}'', \mathcal{M})} \begin{array}{l} C(a) \notin \mathcal{A} \\ \text{or} \\ D(a) \notin \mathcal{A} \end{array} \\
\text{where } \mathcal{A}' = \mathcal{A} \cup \{(C \sqcup D)(a), C(a)\} \\
\mathcal{A}'' = \mathcal{A} \cup \{(C \sqcup D)(a), D(a)\} \\
(\perp_3) \frac{(\mathcal{T}, \mathcal{A}, \mathcal{M})}{\perp} \text{circular}(\mathcal{A}, \mathcal{M}) \\
(\forall) \frac{(\mathcal{T}, \mathcal{A} \cup \{\forall R.C(a), R(a, b)\}, \mathcal{M})}{(\mathcal{T}, \mathcal{A} \cup \{\forall R.C(a), R(a, b), C(b)\}, \mathcal{M})} C(b) \notin \mathcal{A} \\
(\text{trans}') \frac{(\mathcal{T}, \mathcal{A}, \mathcal{M})}{\bigwedge \{(\mathcal{T}, \mathcal{X}_{\exists R.C(a)}) \mid \exists R.C(a) \in \mathcal{A}\}} \\
\text{where } \mathcal{X}_{\exists R.C(a)} = \{C\} \cup \{D \mid \forall R.D(a) \in \mathcal{A}\} \cup \mathcal{T} \\
(\text{close}) \frac{(\mathcal{T}, \mathcal{A}, \mathcal{M})}{(\mathcal{T}, \mathcal{A}', \mathcal{M}') \vee (\mathcal{T}, \mathcal{A}'', \mathcal{M})} \{a, b\} \subseteq \text{dom}(\mathcal{M}) \quad a \neq b \notin \mathcal{A} \quad a \neq b \\
\text{where } \mathcal{A}' = \mathcal{A}[a/b], \mathcal{M}' = \mathcal{M}[a/b] \text{ and } \mathcal{A}'' = \mathcal{A} \cup \{a \neq b\} \\
(=) \frac{(\mathcal{T}, \mathcal{A}, \mathcal{M} \cup \{a =_{\text{m}} A, a =_{\text{m}} B\})}{(\mathcal{T}', \mathcal{A} \cup \mathcal{A}', \mathcal{M} \cup \{a =_{\text{m}} A\})} \\
\text{where } \mathcal{T}' = \mathcal{T} \cup \{(A \sqcup \neg B), (B \sqcup \neg A)\} \\
\mathcal{A}' = \{((A \sqcup \neg B) \sqcap (B \sqcup \neg A))(d) \mid d \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M}) \cup \{a\}\} \\
(\neq) \frac{(\mathcal{T}, \mathcal{A} \cup \{a \neq b\}, \mathcal{M} \cup \{a =_{\text{m}} A, b =_{\text{m}} B\})}{(\mathcal{T}, \mathcal{A}', \mathcal{M} \cup \{a =_{\text{m}} A, b =_{\text{m}} B\})} \text{cond}_{(\neq)} \\
\text{where } \text{cond}_{(\neq)} \text{ means that there is no } d \text{ such that } (A \sqcap \neg B \sqcup \neg A \sqcap B)(d) \in \mathcal{A} \\
\mathcal{A}' = \mathcal{A} \cup \{a \neq b\} \cup \mathcal{A}'' \\
\mathcal{A}'' = \{(A \sqcap \neg B \sqcup \neg A \sqcap B)(d_0)\} \cup \{C(d_0) \mid C \in \mathcal{T}\} \\
\text{for a new individual } d_0
\end{array}$$

**Fig. 7.** Tableau rules for base judgements

and  $\mathcal{A}^*$  and  $\mathcal{M}^*$  are obtained from  $\mathcal{A}$  and  $\mathcal{M}$  by choosing a canonical representative  $a$  for each assertion  $a = b$  and replacing  $b$  by  $a$ .

3. Base nodes are expanded using the rules of Figure 7 while variable nodes are expanded using the rules of Figure 6.
4. For every node  $v$  of the graph, if a  $k$ -ary rule  $\delta$  is applicable to (the label of)  $v$  in the sense that an instance of  $\delta$  has the label of  $v$  as premise and  $Z_1, \dots, Z_k$  as possible conclusions, then choose such a rule according to the preference of Definition 20 and apply it to  $v$  to make  $k$  successors  $w_1, \dots, w_k$  with labels  $Z_1, \dots, Z_k$ , respectively.
5. If the graph already contains a node  $w'_i$  with label  $Z_i$  then instead of creating a new node  $w_i$  with label  $Z_i$  as a successor of  $v$  we just connect  $v$  to  $w'_i$  and assume  $w_i = w'_i$ .
6. If the applied rule is (trans) or (trans') then we label the edge  $(v, w_i)$  by  $\exists R.C$  if the principal formula is either  $\exists R.C$  or  $(\exists R.C)(a)$ .
7. If the rule applied to  $v$  is an or-rule then  $v$  is an or-node. If the rule applied to  $v$  is an and-rule then  $v$  is an and-node.
8. If no rule is applicable to  $v$  then  $v$  is an end-node as well as an and-node.

**Remark 2.** We make the following observations on the construction of the and-or graph.

- The graph cannot contain edges from a variable node to a base node.
- Each non-end node is “expanded” exactly once, using only one rule. Expansion continues until no further expansion is possible.
- The nodes have unique labels.
- Nodes expanded by applying a non-branching static rule can be treated either as or-nodes or and-nodes. We choose to treat them as or-nodes. Applying the  $(\sqcap)$  to a node causes the node to become an or-node (which might seem counter-intuitive).
- The graph is finite (see Lemma 13).

**Definition 22 (Marking).** A marking of an and-or graph  $\mathbb{G}$  is a subgraph  $\mathbb{G}'$  of  $\mathbb{G}$  such that:

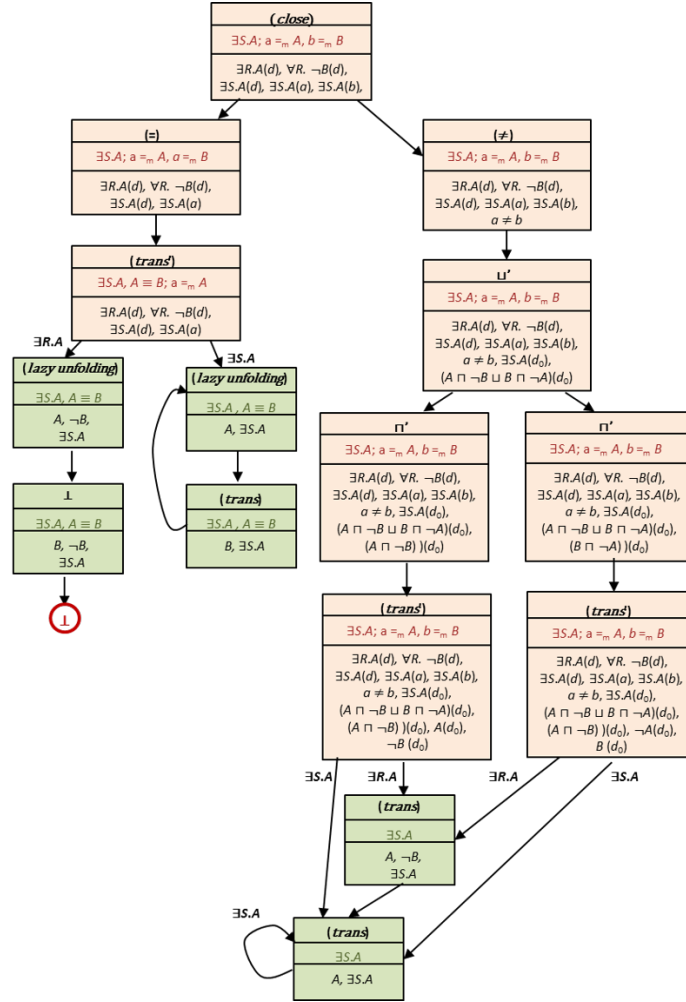
- the root of  $\mathbb{G}$  is the root of  $\mathbb{G}'$ .
- if  $v$  is a node of  $\mathbb{G}'$  and is an or-node of  $\mathbb{G}$  then there exists exactly one edge  $(v, w)$  of  $\mathbb{G}$  that is an edge of  $\mathbb{G}'$ .
- if  $v$  is a node of  $\mathbb{G}'$  and is an and-node of  $\mathbb{G}$  then every edge  $(v, w)$  of  $\mathbb{G}$  is an edge of  $\mathbb{G}'$ .
- if  $(v, w)$  is an edge of  $\mathbb{G}'$  then  $v$  and  $w$  are nodes of  $\mathbb{G}'$ .

A marking  $\mathbb{G}'$  of an and-or graph  $\mathbb{G}$  for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  is consistent if it does not contain any node with label  $\perp$ .

**Example 4.** Figure 8 shows the and-or graph for the following knowledge base.

Tbox	Abox	Mbox
$\top \sqsubseteq \exists S.A$	$\exists R.A(d) \quad \forall R.\neg B(d)$	$a =_m A \quad b =_m B$

To reduce the number of nodes, we apply lazy unfolding (denoted by *l. unf.* in the figure) [23]. Instead of adding  $(A \sqcup \neg B) \sqcap (B \sqcup \neg A)$  in the  $(=)$ -rule, we add  $A \equiv B$ . Then, we do lazy unfolding and replace  $A$  by  $B$  in expressions of the form  $A$  or  $\neg A$  that appear in  $\mathcal{X}$  or in expressions of the form  $A(x)$  or  $\neg A(x)$  that appear in  $\mathcal{A}$ .



**Fig. 8.** And-or graph for the knowledge base:  $\top \sqsubseteq \exists S.A, \exists R.A(d), \forall R. \neg B(d), a =_m A, b =_m B$

## 7 Correctness of the Tableau Calculus for $\mathcal{ALCM}$

In this section we prove correctness of the Tableau Calculus for  $\mathcal{ALCM}$  presented in the previous section.

**Theorem 2 (Correctness of the Tableau Calculus).** *The  $\mathcal{ALCM}$  knowledge base  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  is consistent if and only if the and-or graph for  $\mathcal{K}$  has a consistent marking.*

The if direction is proved in Theorem 3 (Soundness) and the only if direction is proved in Theorem 9 (Completeness). We give an idea of the proof. For the if direction, it is necessary to prove that the rules preserve satisfiability. A consistent marking exists because we start from a satisfiable node with label  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$ . For the only if direction, we prove that the converse of base rules preserve satisfiability, i.e. if the conclusion is satisfiable, so is the premise. Then, it is enough to construct a canonical model for a base node where no base rules are applicable. We know at that point that the Mbox has no circularities which is crucial to enforce that the model is well-founded. We also know that the  $(=)$  and  $(\neq)$ -rules cannot longer be applied, which means that there is an isomorphism between the canonical interpretation of  $\mathcal{ALC}$  and  $\mathcal{ALCM}$ .

We illustrate the idea of canonical interpretation with the following example. Suppose we have an ontology  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  with four individuals  $a, b, c$  and  $d$  with axioms  $B(a)$ ,  $A(c)$ ,  $A(d)$  and the meta-modelling axioms given by  $a =_m A$  and  $b =_m B$ . The canonical interpretation  $\mathcal{I}$  of the  $\mathcal{ALC}$  ontology is then,

$$\begin{aligned}\Delta^{\mathcal{I}} &= \{a, b, c, d\} \\ A^{\mathcal{I}} &= \{c, d\} \\ B^{\mathcal{I}} &= \{a\}\end{aligned}$$

Intuitively, we see that we need to force the following equations to make the meta-modelling axioms  $a =_m A$  and  $b =_m B$  satisfiable:

$$\begin{aligned}a &= \{c, d\} \\ b &= \{a\}\end{aligned}$$

These equations do not have circularities. Then, the canonical interpretation  $\mathcal{I}'$  for the ontology in  $\mathcal{ALCM}$  is now defined as follows.

$$\begin{aligned}\Delta^{\mathcal{I}'} &= \{\{c, d\}, \{\{c, d\}\}, c, d\} \\ A^{\mathcal{I}'} = a^{\mathcal{I}'} &= \{c, d\} \\ B^{\mathcal{I}'} = b^{\mathcal{I}'} &= \{\{c, d\}\}\end{aligned}$$

In this case,  $\mathcal{I}'$  is a model of  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ . By defining  $S_0 = \{c, d\}$ , we see that  $\Delta^{\mathcal{I}'} \subset S_2$ .

### 7.1 Soundness

In order to prove soundness, the following lemma is crucial:

**Lemma 6 (Preservation of Satisfiability in the Tableau Calculus).** *All the rules of Figures 6 and 7 preserve satisfiability, i.e. if the premise is satisfiable so is the conclusion.*

*Proof.* Note that the statement holds trivially for the bottom rules since their premises are unsatisfiable. We now give the proof of the statement for some of the rules that are most interesting.

(close)-**rule.** Suppose the premise  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  is satisfiable. Then, there exists an interpretation  $\mathcal{I}$  such that

- $\mathcal{I} \models \bigwedge \mathcal{T} \equiv \top$
- $\mathcal{I} \models \mathcal{A}$
- $\mathcal{I} \models \mathcal{M}$

According to the interpretation  $\mathcal{I}$ , given two individuals  $a, b$  belonging to  $\text{dom}(\mathcal{M})$  we have that either  $a^{\mathcal{I}} = b^{\mathcal{I}}$  or  $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ .

We will prove that in both cases  $(\mathcal{T}, \mathcal{A}[a/b], \mathcal{M}[a/b]) \vee (\mathcal{T}, \mathcal{A} \cup \{a \neq b\}, \mathcal{M})$  is satisfiable.

1. Suppose  $a^{\mathcal{I}} = b^{\mathcal{I}}$ . It is easy to prove the following properties:

$$\mathcal{I} \models \mathcal{A}, a^{\mathcal{I}} = b^{\mathcal{I}} \text{ implies } \mathcal{I} \models \mathcal{A}[a/b]$$

$$\mathcal{I} \models \mathcal{M}, a^{\mathcal{I}} = b^{\mathcal{I}} \text{ implies } \mathcal{I} \models \mathcal{M}[a/b]$$

Hence,  $(\mathcal{T}, \mathcal{A}[a/b], \mathcal{M}[a/b])$  is satisfiable and so is  $(\mathcal{T}, \mathcal{A}[a/b], \mathcal{M}[a/b]) \vee (\mathcal{T}, \mathcal{A} \cup \{a \neq b\}, \mathcal{M})$ .

2. Suppose  $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ . This means that  $\mathcal{I} \models a \neq b$  and then  $\mathcal{I} \models \mathcal{A} \cup \{a \neq b\}$ . Therefore,  $(\mathcal{T}, \mathcal{A} \cup \{a \neq b\}, \mathcal{M})$  is satisfiable and so is  $(\mathcal{T}, \mathcal{A}[a/b], \mathcal{M}[a/b]) \vee (\mathcal{T}, \mathcal{A} \cup \{a \neq b\}, \mathcal{M})$ .

(=)-**rule.** Suppose the premise  $(\mathcal{T}, \mathcal{A}, \mathcal{M} \cup \{a =_{\text{m}} A, a =_{\text{m}} B\})$  is satisfiable. Hence, there exists an interpretation  $\mathcal{I}$  such that

- $\mathcal{I} \models \bigwedge \mathcal{T} \equiv \top$
- $\mathcal{I} \models \mathcal{A}$
- $\mathcal{I} \models \mathcal{M} \cup \{a =_{\text{m}} A, a =_{\text{m}} B\}$

We need to prove that:

1.  $\mathcal{I} \models \bigwedge (\mathcal{T} \cup \{(A \sqcup \neg B) \sqcap (B \sqcup \neg A)\}) \equiv \top$
2.  $\mathcal{I} \models \mathcal{A} \cup \{((A \sqcup \neg B) \sqcap (B \sqcup \neg A))(d) \mid d \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M}) \cup \{a\}\}$
3.  $\mathcal{I} \models \mathcal{M} \cup \{a =_{\text{m}} A\}$

1. Since  $\mathcal{I} \models a =_{\text{m}} A$ , we have that

$$a^{\mathcal{I}} = A^{\mathcal{I}} \tag{1}$$

and since  $\mathcal{I} \models a =_m B$ , we also have that

$$a^{\mathcal{I}} = B^{\mathcal{I}} \quad (2)$$

From (1) and (2), we get

$$A^{\mathcal{I}} = B^{\mathcal{I}} \quad (3)$$

We will prove that  $\mathcal{I}$  validates  $(A \sqcup \neg B) \cap (B \sqcup \neg A)$ , that is

$$((A \sqcup \neg B) \cap (B \sqcup \neg A))^{\mathcal{I}} = \Delta^{\mathcal{I}}.$$

Applying the definition of interpretation we have that

$$((A \sqcup \neg B) \cap (B \sqcup \neg A))^{\mathcal{I}} = (A^{\mathcal{I}} \cup (\Delta \setminus B^{\mathcal{I}})) \cap (B^{\mathcal{I}} \cup (\Delta \setminus A^{\mathcal{I}}))$$

From (3) and replacing in the above expression we obtain

$$((A \sqcup \neg B) \cap (B \sqcup \neg A))^{\mathcal{I}} = (A^{\mathcal{I}} \cup (\Delta \setminus A^{\mathcal{I}})) \cap (A^{\mathcal{I}} \cup (\Delta \setminus A^{\mathcal{I}})) = \Delta^{\mathcal{I}}$$

So,  $\mathcal{I} \models (A \sqcup \neg B) \cap (B \sqcup \neg A) \equiv \top$  and then

$$\mathcal{I} \models \bigcap (\mathcal{T} \cup \{(A \sqcup \neg B) \cap (B \sqcup \neg A)\}) \equiv \top.$$

2. Since  $((A \sqcup \neg B) \cap (B \sqcup \neg A))^{\mathcal{I}} = \Delta^{\mathcal{I}}$ , we have that

$$d^{\mathcal{I}} \in ((A \sqcup \neg B) \cap (B \sqcup \neg A))^{\mathcal{I}}$$

for all  $d \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .

3.  $\mathcal{M} \cup \{a =_m A\} \subset \mathcal{M} \cup \{a =_m A, a =_m B\}$  and  
 $\mathcal{I} \models \mathcal{M} \cup \{a =_m A, a =_m B\}$ , so  $\mathcal{I} \models \mathcal{M} \cup \{a =_m A\}$

This means that the conclusion of the  $(=)$ -rule

$$(\mathcal{T}_1, \mathcal{A}_1, \mathcal{M} \cup \{a =_m A\})$$

where  $\mathcal{T}_1 := \mathcal{T} \cup \{(A \sqcup \neg B), (B \sqcup \neg A)\}$  and  $\mathcal{A}_1 = \mathcal{A} \cup \{((A \sqcup \neg B) \cap (B \sqcup \neg A))(d) \mid d \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M}) \cup \{a\}\}$  is satisfiable.

**( $\neq$ )-rule.** Suppose the premise  $(\mathcal{T}, \mathcal{A} \cup \{a \neq b\}, \mathcal{M} \cup \{a =_m A, b =_m B\})$  is satisfiable. Consequently, there exists an interpretation  $\mathcal{I}$  such that

- $\mathcal{I} \models \bigcap \mathcal{T} \equiv \top$
- $\mathcal{I} \models \mathcal{A}, a \neq b$                       thus  $\mathcal{I} \models \mathcal{A}$  and  $\mathcal{I} \models a \neq b$
- $\mathcal{I} \models \mathcal{M}, a =_m A, b =_m B$  thus  $\mathcal{I} \models \mathcal{M}, \mathcal{I} \models a =_m A$  and  $\mathcal{I} \models b =_m B$

In order to prove that the conclusion of the rule is satisfiable we only need to prove that

1.  $\mathcal{I} \models (A \sqcap \neg B \sqcup \neg A \sqcap B)(d_0)$  where  $d_0$  is a new individual and
  2.  $\mathcal{I} \models C(d_0)$  for all  $C \in \mathcal{T}$
- as the other conditions follow immediately from the satisfiable property of the premise.

1. Since  $\mathcal{I} \models a \neq b$ , we have that

$$a^{\mathcal{I}} \neq b^{\mathcal{I}} \quad (4)$$

Since  $\mathcal{I} \models a =_{\text{m}} A$  and  $\mathcal{I} \models b =_{\text{m}} B$ , we also have that

$$a^{\mathcal{I}} = A^{\mathcal{I}} \quad b^{\mathcal{I}} = B^{\mathcal{I}} \quad (5)$$

It follows from (4) and (5) that the sets  $A^{\mathcal{I}}$  and  $B^{\mathcal{I}}$  are different. Hence, there is an element in one of these sets that is not in the other.

That is to say, there exists an element on the domain of the interpretation ( $x \in \Delta^{\mathcal{I}}$ ) that belongs to the set  $(A^{\mathcal{I}} \cap (\Delta \setminus B^{\mathcal{I}})) \cup ((\Delta \setminus A^{\mathcal{I}}) \cap B^{\mathcal{I}})$ .

If we use  $d_0$ , a new individual in  $\mathcal{K}$ , to denote  $x$  ( $d_0^{\mathcal{I}} = x$ ) we have that  $\mathcal{I} \models (A \sqcap \neg B \sqcup \neg A \sqcap B)(d_0)$ .

2. Since  $\mathcal{I} \models \bigwedge \mathcal{T} \equiv \top$  we have that  $\mathcal{I}$  validates every concept  $C \in \mathcal{T}$ . This means that  $x \in C^{\mathcal{I}}$  for all  $x \in \Delta^{\mathcal{I}}$  and for all  $C \in \mathcal{T}$ . In particular:  $d_0^{\mathcal{I}} \in C^{\mathcal{I}}$  and hence,  $\mathcal{I} \models C(d_0)$  for all  $C \in \mathcal{T}$ .

In this way we can conclude that the  $(\neq)$ -rule preserves satisfiability.

**Lemma 7 (Preservation of Satisfiability in the And-or-graph).** *In an And-or graph  $\mathbb{G}$ , for every no-end node with satisfiable label:*

- if it is an and-node, all its successors have a satisfiable label.
- if it is an or-node, at least one of its successors has a satisfiable label.

*Proof.* Let  $v$  be a no-end node with label  $E_1$  satisfiable. The successors of  $v$  are obtained by applying one of the rules of Figures 6 and 7 (let it be  $r$ ) using  $E_1$  as premise of the rule. In addition, the labels of the successors nodes are obtained from the conclusion of the rule.

**$v$  is an and-node** The conclusion of  $r$  is a and-judgement of the form:  $J_1 \wedge \dots \wedge J_k$ , where  $J_i$  are simple judgements for all  $1 \leq i \leq k$ .

Applying Lemma 6 we know that  $J_1 \wedge \dots \wedge J_k$  is satisfiable since  $E_1$  (premise of  $r$ ) is satisfiable. So each  $J_i$  is satisfiable. Based on the construction of the graph  $\mathbb{G}$  the labels of the successors of  $v$  are each  $J_i$ . Consequently, the labels of all the successors of  $v$  are satisfiable.

**$v$  is an or-node** The conclusion of  $r$  is a or-judgement of the form:  $J_1 \vee J_2$ , where  $J_1$  and  $J_2$  are simple judgements. Since  $E_1$  (premise of  $r$ ) is satisfiable, so is  $J_1 \vee J_2$  (applying Lemma 6). From the definition of satisfiable we know that  $J_1$  is satisfiable or  $J_2$  is satisfiable.

In these cases the node  $v$  has two successors  $w_1, w_2$  and the labels of these nodes are  $J_1$  and  $J_2$  respectively. Hence, at least one of the successors of  $v$  has a label which is satisfiable.

**Theorem 3 (Soundness of the Tableau Calculus of  $\mathcal{ALCCM}$ ).** *Let  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  be a knowledge base of  $\mathcal{ALCCM}$  in negation normal form. If  $\mathcal{K}$  is consistent then the and-or graph for  $\mathcal{K}$  has a consistent marking.*

*Proof.* Let  $\mathbb{G}$  be the and-or graph for  $\mathcal{K}$ . We will construct  $\mathbb{G}'$  a consistent marking of  $\mathbb{G}$ .

- First of all, we initialize  $\mathbb{G}'$  with the root of  $\mathbb{G}$ . If  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  is consistent then the label of this node calculated as stated in definition 21 is satisfiable.
- Then for each node  $v$  in  $\mathbb{G}'$ :
  - if  $v$  is an and-node we add all the successors of  $v$  in  $\mathbb{G}$  to  $\mathbb{G}'$ , and all the edges between them. Applying Lemma 7, we deduce that all these new nodes in  $\mathbb{G}'$  have satisfiable labels.
  - if  $v$  is an or-node we add to  $\mathbb{G}'$  the successors of  $v$  in  $\mathbb{G}$  having a satisfiable label. We know this node exists from Lemma 7.

It is easy to show that  $\mathbb{G}'$  is a marking of  $\mathbb{G}$  (see Definition 22). The node we use to initialize  $\mathbb{G}'$  has a satisfiable label and we ensure that each node we add has a satisfiable label, so in  $\mathbb{G}'$  there is no node with label  $\perp$ . Consequently  $\mathbb{G}'$  is a consistent marking of  $\mathbb{G}$ . Since  $\mathbb{G}'$  is a subgraph of a finite graph (see Remark 2), it is finite.

## 7.2 Completeness

We define three notions of saturated **R**-structure 1) for a Tbox  $\mathcal{T}$ , 2) for an  $\mathcal{ALC}$ -knowledge base  $(\mathcal{T}, \mathcal{A})$  and 3) for an  $\mathcal{ALCCM}$ -knowledge base  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ . These three notions are conditions that ensure satisfiability of a set  $\mathcal{X}$  of concepts w.r.t.  $\mathcal{T}$ , satisfiability of a  $\mathcal{ALC}$ -knowledge base  $(\mathcal{T}, \mathcal{A})$  and satisfiability of an  $\mathcal{ALCCM}$ -knowledge base  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  respectively (see Theorems 4, 5 and 7). They correspond to the so-called “tableau” in Description Logic [24,1]. A saturated **R**-structure can be seen as an abstraction of a model. The weaker notion of consistent model graph given by Nguyen and Szalas [15] includes clauses 1-5 of Definition 24 but not the last one. Hence, it only ensures satisfiability of a set of concepts with respect to an empty Tbox. The sufficient conditions for satisfiability (Theorems 4, 5 and 7) structure the proof of completeness and make it more neat.

**Definition 23 (R-structure).** *We say that  $(\Delta, \mathcal{L}, \mathcal{E})$  is an **R**-structure if*

- $\Delta$  is a non-empty set,
- $\mathcal{L}$  maps each element in  $\Delta$  to a set of concepts,
- $\mathcal{E} : \mathbf{R} \rightarrow 2^{\Delta \times \Delta}$  maps each role in  $\mathbf{R}$  to a set of pairs of elements in  $\Delta$ .

**Definition 24 (Saturated R-structure).** *We say that  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  is a saturated **R**-structure for a Tbox  $\mathcal{T}$  if  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  is an **R**-structure that satisfies the following properties for all  $x, y \in \Delta$ ,  $R \in \mathbf{R}$  and concepts  $C, C_1, C_2$ .*

1. If  $\neg C \in \mathcal{L}(x)$ , then  $C \notin \mathcal{L}(x)$ .
2. If  $C_1 \sqcap C_2 \in \mathcal{L}(x)$ , then  $C_1 \in \mathcal{L}(x)$  and  $C_2 \in \mathcal{L}(x)$ .
3. If  $C_1 \sqcup C_2 \in \mathcal{L}(x)$ , then  $C_1 \in \mathcal{L}(x)$  or  $C_2 \in \mathcal{L}(x)$ .



4. If  $\forall R.C \in \mathcal{L}(x)$  and  $(x, y) \in \mathcal{E}(R)$ , then  $C \in \mathcal{L}(y)$ .
5. If  $\exists R.C \in \mathcal{L}(x)$ , then there is some  $y \in \Delta$  such that  $(x, y) \in \mathcal{E}(R)$  and  $C \in \mathcal{L}(y)$ .
6. If  $C \in \mathcal{T}$  then  $C \in \mathcal{L}(x)$  for all  $x \in \Delta$ .

**Definition 25.** The interpretation  $\mathcal{I}$  induced by an **R**-structure is defined as follows.

$$\begin{aligned}\Delta^{\mathcal{I}} &= \Delta \\ A^{\mathcal{I}} &:= \{x \in \Delta \mid A \in \mathcal{L}(x)\} \\ R^{\mathcal{I}} &:= \mathcal{E}(R)\end{aligned}$$

As usual, it is enough for an interpretation as the one given above to define it only for atomic concepts  $A$ . Next lemma gives a characterization of that interpretation for complex concepts.

**Lemma 8.** Let  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  be a saturated **R**-structure for a Tbox  $\mathcal{T}$  and  $\mathcal{I}$  be the interpretation induced by  $\mathbb{S}$ . For every  $x \in \Delta$ , if  $C \in \mathcal{L}(x)$  then  $x \in C^{\mathcal{I}}$ .

*Proof.* This is proved by induction on  $C$ . We prove some cases.

- Suppose  $C = A$ . This is the base case. By Definition 25, we have that  $x \in A^{\mathcal{I}}$ .
- Suppose  $C = \neg D$ . Since  $C$  is in negation normal form, we have that  $D$  is an atomic concept, say  $A$ . Since  $\mathbb{S}$  is a saturated **R**-structure, by Definition 24,  $A \notin \mathcal{L}(x)$ . By Definition 25, we have that  $x \notin A^{\mathcal{I}}$ . Hence,  $x \in (\neg A)^{\mathcal{I}}$ .
- Suppose  $C = C_1 \sqcap C_2$ . By Definition 24,  $C_1 \in \mathcal{L}(x)$  and  $C_2 \in \mathcal{L}(x)$ . By induction hypothesis we have that  $x \in C_1^{\mathcal{I}}$  and  $x \in C_2^{\mathcal{I}}$ . Then  $x \in C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$ , so  $x \in (C_1 \sqcap C_2)^{\mathcal{I}}$ , which means that  $x \in C^{\mathcal{I}}$ .
- Suppose  $C = \exists R.D$ . By Definition 24, there is some  $y \in \Delta$  such that  $(x, y) \in \mathcal{E}(R)$  and  $D \in \mathcal{L}(y)$ . By Definition 25, we have that  $(x, y) \in R^{\mathcal{I}}$ . By induction hypothesis,  $y \in D^{\mathcal{I}}$ . Then  $x \in (\exists R.D)^{\mathcal{I}}$ , and so  $x \in C^{\mathcal{I}}$ .

**Theorem 4.** Let  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  be a saturated **R**-structure for a Tbox  $\mathcal{T}$ . If  $\mathcal{X} \subseteq \mathcal{L}(x_0)$  for some  $x_0 \in \Delta$  then  $\mathcal{X}$  is satisfiable w.r.t.  $\mathcal{T}$ .

*Proof.* Let  $\mathcal{I}$  be an interpretation induced by  $\mathbb{S}$ . It follows from Lemma 8 that  $x_0 \in C^{\mathcal{I}}$  for all  $C \in \mathcal{X}$ . Hence  $(\bigcap \mathcal{X})^{\mathcal{I}} \neq \emptyset$ . We also have that  $\mathcal{I}$  is a model of  $\mathcal{T}$  from the last clause of Definition 24.

**Definition 26 (Saturated **R**-structure for  $(\mathcal{T}, \mathcal{A})$ ).** We say that  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  is a saturated **R**-structure for  $(\mathcal{T}, \mathcal{A})$  if  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  is a saturated **R**-structure for  $\mathcal{T}$  that satisfies the following properties.

1.  $\Delta$  contains all the individuals of  $\mathcal{A}$ .
2. If  $C(a) \in \mathcal{A}$ , then  $C \in \mathcal{L}(a)$ .
3. If  $R(a, b) \in \mathcal{A}$ , then  $(a, b) \in \mathcal{E}(R)$ .
4. If  $a \neq b \in \mathcal{A}$  then  $a$  and  $b$  are syntactically different.

**Theorem 5.** Let  $\mathcal{A}$  an Abox without equality axioms and  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  be a saturated **R**-structure for  $(\mathcal{T}, \mathcal{A})$ . Then, the  $\mathcal{ALC}$ -knowledge base  $(\mathcal{T}, \mathcal{A})$  is satisfiable (consistent).

*Proof.* Let  $\mathcal{I}$  be the interpretation induced by  $\mathbb{S}$ . For each individual  $a$  of  $\mathcal{A}$ , we define  $a^{\mathcal{I}} = a$ . We will show that  $\mathcal{I}$  is a model for  $(\mathcal{T}, \mathcal{A})$ .

$\mathcal{I}$  is a model of  $\mathcal{T}$  by the last clause of Definition 24 and Lemma 8. To show that  $\mathcal{I}$  is a model of  $\mathcal{A}$ , we will prove that  $a^{\mathcal{I}} \in C^{\mathcal{I}}$  for all  $C(a) \in \mathcal{A}$ ,  $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$  for all  $R(a, b) \in \mathcal{A}$  and  $a^{\mathcal{I}} \neq b^{\mathcal{I}}$  for all  $a \neq b \in \mathcal{A}$ .

- Suppose  $C(a) \in \mathcal{A}$ . By Definition 26,  $C \in \mathcal{L}(a)$  and by Lemma 8,  $a^{\mathcal{I}} = a \in C^{\mathcal{I}}$ .
- Suppose  $R(a, b) \in \mathcal{A}$ . By Definition 26,  $(a, b) \in \mathcal{E}(R)$  and by Definition 25,  $(a^{\mathcal{I}}, b^{\mathcal{I}}) = (a, b) \in R^{\mathcal{I}}$ .
- Suppose  $a \neq b \in \mathcal{A}$ . By Definition 26,  $a$  and  $b$  are syntactically different, so  $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ .

**Definition 27 (Circularity of an  $\mathbf{R}$ -structure w.r.t an Mbox).** We say that  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  has a circularity w.r.t.  $\mathcal{M}$  if there is a sequence of meta-modelling axioms  $a_1 =_{\mathbf{m}} A_1, a_2 =_{\mathbf{m}} A_2, \dots, a_n =_{\mathbf{m}} A_n$  all in  $\mathcal{M}$  such that  $A_1 \in \mathcal{L}(a_2)$ ,  $A_2 \in \mathcal{L}(a_3)$ ,  $\dots$ ,  $A_n \in \mathcal{L}(a_1)$ .

**Definition 28 (Saturated  $\mathbf{R}$ -structure for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ ).** We say that  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  is a saturated  $\mathbf{R}$ -structure for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  if  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  is a saturated  $\mathbf{R}$ -structure for  $(\mathcal{T}, \mathcal{A})$  that satisfies the following properties.

1.  $\Delta$  contains all the individuals of  $\mathcal{M}$ .
2.  $\mathbb{S}$  has no circularities w.r.t.  $\mathcal{M}$ .
3. if  $a =_{\mathbf{m}} A \in \mathcal{M}$  and  $a =_{\mathbf{m}} B \in \mathcal{M}$ , then  $A = B$ , i.e.  $A$  and  $B$  are syntactically equal.
4. if  $a$  and  $b$  are syntactically different,  $a =_{\mathbf{m}} A \in \mathcal{M}$  and  $b =_{\mathbf{m}} B \in \mathcal{M}$ , then there is some  $t \in \Delta$  such that  $A \sqcap \neg B \sqcup B \sqcap \neg A \in \mathcal{L}(t)$ .

Note that the set  $\Delta$  of a saturated  $\mathbf{R}$ -structure for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  contains the individuals of  $\mathcal{M}$  as well as the ones of  $\mathcal{A}$ . In Theorem 8 we construct a  $\mathbf{R}$ -structure for a knowledge base  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  where no other rule is applicable to  $\mathcal{K}$  except for the (trans')-rule. In particular,  $\mathcal{A}$  does not contain equalities and  $\mathcal{M}$  does not contain two axioms with the same individual  $a$ , i.e. if  $a =_{\mathbf{m}} A$  and  $a =_{\mathbf{m}} B$  are in  $\mathcal{M}$  then  $A$  and  $B$  are syntactically equal.

**Definition 29 (From Basic Objects to Sets).** Let  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  be a saturated  $\mathbf{R}$ -structure for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ . For  $x \in \Delta$  we define  $\text{set}(x)$  as follows.

$$\begin{aligned} \text{set}(x) &= x && \text{if } x \notin \text{dom}(\mathcal{M}) \\ \text{set}(x) &= \{\text{set}(y) \mid A \in \mathcal{L}(y)\} && \text{otherwise, i.e. } x \in \text{dom}(\mathcal{M}) \text{ and } x =_{\mathbf{m}} A \in \mathcal{M} \end{aligned}$$

Since  $\Delta$  contains the individuals of  $\mathcal{M}$ , an element of  $\Delta$  either belongs to  $\text{dom}(\mathcal{M})$  or not. In case,  $x \in \text{dom}(\mathcal{M})$  then we have that  $x =_{\mathbf{m}} A \in \mathcal{M}$ .

*Example 5.* We consider the example of Figure 4 and add a new individual *hydrographic* and the meta-modelling axiom

$$\text{hydrographic} =_{\mathbf{m}} \text{HydrographicObject}$$

Here we have for example that *river* is an individual with meta-modelling. As such, its interpretation should be a set and not a basic object. The set associated to *river* is given by the function *set* and it is as follows.

$$\text{set}(\text{river}) = \{\text{queguay}, \text{santaLucia}\}$$

The individual *hydrographic* has also meta-modelling. But its inhabitants also have meta-modelling. The set associated to *hydrographic* is a set of sets given as follows.

$$\text{set}(\text{hydrographic}) = \{\{\text{queguay}, \text{santaLucia}\}, \{\text{deRocha}, \text{delSauce}\}\}$$

On the other hand, *queguay* does not have meta-modelling and we define *set* as follows.

$$\text{set}(\text{queguay}) = \text{queguay}.$$

**Theorem 6 (Correctness of the recursive definition).** *Let  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  be a saturated  $\mathbf{R}$ -structure for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ . The function *set* is a correct recursive definition.*

*Proof.* It follows from the third clause of Definition 28 that *set* is indeed a function. In order to prove that *set* is a correct recursive definition, we apply the recursion principle on well-founded relations (see Definition 6).

For this, we define the relation  $\prec$  on  $\Delta$  as  $y \prec a$  iff  $A \in \mathcal{L}(y)$  and  $a =_m A \in \mathcal{M}$ . Since  $\mathcal{L}$  has no circularities w.r.t.  $\mathcal{M}$ , it is easy to prove that  $\prec$  is well-founded. Suppose towards a contradiction that  $\prec$  is not well-founded. It follows from Lemma 2, that there exists an infinite  $\prec$ -decreasing sequence, i.e. there is  $\langle x_i \rangle_{i \in \mathbb{N}}$  such that  $x_{i+1} \prec x_i$  and  $x_i \in \Delta$  for all  $i \in \mathbb{N}$ .

$$\dots \prec x_{i+1} \prec x_i \prec \dots \prec x_1$$

By definition of  $\prec$ , we have that  $x_i \in \text{dom}(\mathcal{M})$  for all  $i \in \mathbb{N}$ . Since the Mbox is finite, there exists an element in the above sequence that should occur at least twice, i.e.  $x_{i+1} = x_{i+n+1} = a_1$  for some  $i, n \in \mathbb{N}$ . Hence, we have a cycle

$$\begin{array}{ccccccccc} x_{i+n+1} & \prec & x_{i+n} & \prec & x_{n+i-1} & \prec & \dots & \prec & x_{i+2} & \prec & x_{i+1} \\ \parallel & & \parallel & & \parallel & & & & \parallel & & \parallel \\ a_1 & & a_n & & a_{n-1} & & & & a_2 & & a_1 \end{array}$$

It is easy to see that this contradicts the fact that  $\mathbb{S}$  has no circularities w.r.t.  $\mathcal{M}$ .

Since  $\prec$  is well-founded, we can apply the recursion principle in Definition 29. Note that in the recursive step of that definition, we have that  $y \prec x$ .

**Lemma 9.** *Let  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  be a saturated  $\mathbf{R}$ -structure for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ . If  $S_0 = \{x \in \Delta \mid x \notin \text{dom}(\mathcal{M})\}$  then*

$$\text{set}(x) \in S_{\# \mathcal{M}}$$

*Proof.* Let  $\text{maxl}^\prec(x)$  be the *maximal length* of all descending  $\prec$ -sequences starting from  $x \in \Delta$ . This number is finite because  $\prec$  is well-founded by Theorem 6. It is not difficult to show that

$$\text{maxl}^\prec(x) \leq \sharp \mathcal{M} \text{ for all } x \text{ in } \Delta.$$

It is also easy to prove that

$$\text{set}(x) \in S_{\text{maxl}^\prec(x)}$$

by induction on  $\text{maxl}^\prec(x)$ . Then  $\text{set}(x) \in S_{\text{maxl}^\prec(x)} \subseteq S_{\sharp \mathcal{M}}$  since  $S_n$  is a monotonic function on  $n \in \mathbb{N}$  (see Definition 9).

**Lemma 10.** *Let  $\mathcal{A}$  be an Abox without equality axioms,  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  be a saturated **R**-structure for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ ,  $\{a =_m A, b =_m B\} \subseteq \mathcal{M}$  and  $\mathcal{I}$  the interpretation induced by  $\mathbb{S}$ . Then,  $a$  and  $b$  are syntactically equal if and only if  $A^\mathcal{I} = B^\mathcal{I}$ .*

*Proof.* – if  $a$  and  $b$  are syntactically equal then  $a =_m A$  and  $a =_m B$  are both in  $\mathcal{M}$ . Applying the Definition 28 we have that  $A = B$  and so are their interpretations.

- if  $a$  and  $b$  are syntactically different, applying Definition 28, it follows that there is some  $t \in \Delta$  such that  $A \sqcap \neg B \sqcup B \sqcap \neg A \in \mathcal{L}(t)$ . Applying Definition 24 we have two possibles cases:
  1.  $A \in \mathcal{L}(t)$  and  $B \notin \mathcal{L}(t)$ , so  $t \in A^\mathcal{I}$  and  $t \notin B^\mathcal{I}$ , then  $A^\mathcal{I} \not\subseteq B^\mathcal{I}$ .
  2.  $A \notin \mathcal{L}(t)$  and  $B \in \mathcal{L}(t)$ , so  $t \in B^\mathcal{I}$  and  $t \notin A^\mathcal{I}$ , then  $B^\mathcal{I} \not\subseteq A^\mathcal{I}$ .

In both cases,  $A^\mathcal{I} \neq B^\mathcal{I}$

**Theorem 7.** *Let  $\mathcal{A}$  an Abox without equality axioms and  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  be a saturated **R**-structure for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ . Then,  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  is satisfiable (consistent).*

*Proof.* The interpretation  $\mathcal{I}$  induced by  $\mathbb{S}$  is a model of  $(\mathcal{T}, \mathcal{A})$  by Theorem 5. We define the interpretation  $\mathcal{I}'$  as follows.

$$\begin{aligned} \Delta^{\mathcal{I}'} &= \{\text{set}(x) \mid x \in \Delta^\mathcal{I}\} \\ A^{\mathcal{I}'} &= \{\text{set}(x) \mid x \in A^\mathcal{I}\} \\ R^{\mathcal{I}'} &= \{(\text{set}(x), \text{set}(y)) \mid (x, y) \in R^\mathcal{I}\} \\ a^{\mathcal{I}'} &= \text{set}(a^\mathcal{I}) \end{aligned}$$

We prove the three clauses of Definition 10.

1. It follows from Lemma 9 that

$$\Delta^{\mathcal{I}'} \subseteq S_{\sharp \mathcal{M}}$$

2. By Lemma 1,  $\mathcal{I}'$  is a model of  $(\mathcal{T}, \mathcal{A})$  because  $\mathcal{I}$  and  $\mathcal{I}'$  are isomorphic interpretations.

**Claim.**  $\mathcal{I}$  and  $\mathcal{I}'$  are isomorphic interpretations.

In order to prove the claim, we prove that  $\text{set}$  is a bijection between the domains of  $\mathcal{I}$  and  $\mathcal{I}'$  as follows.

- It follows from Theorem 6 that  $\text{set}$  is a function.
- It is surjective since the elements of the domain of  $\mathcal{I}'$  are defined as the result of applying  $\text{set}$  to the elements of the domain of  $\mathcal{I}$ .

- We prove that  $\text{set}$  is injective by induction on  $\prec$  by applying the Induction Principle given in Definition 4. We prove that if  $x_1$  and  $x_2$  are two different elements in  $\Delta$  then  $\text{set}(x_1) \neq \text{set}(x_2)$ , or equivalently that if  $\text{set}(x_1) = \text{set}(x_2)$  then  $x_1 = x_2$ . Suppose now that  $x_1$  and  $x_2$  are two different elements of  $\Delta$ .
  - **Base Case.** Suppose  $x_1, x_2$  are individuals without meta-modelling. It follows from the definition of  $\text{set}$  that  $\text{set}(x_1) \neq \text{set}(x_2)$ .
  - **Base Case.** Suppose only one of them has meta-modelling, say  $x_1$ . We know that  $\text{set}(x_1) \notin \Delta$  since it is a set but  $\text{set}(x_2) \in \Delta$  (is  $x_2$ ) so  $\text{set}(x_1) \neq \text{set}(x_2)$ .
  - **Inductive Case.** Suppose that  $x_1$  and  $x_2$  are individuals with meta-modelling:  $x_1 =_{\mathcal{M}} A_1$  and  $x_2 =_{\mathcal{M}} A_2$ . It follows from the definition of  $\text{set}$  that:
 
$$\text{set}(x_1) = \{\text{set}(q) \mid A_1 \in \mathcal{L}(q)\}$$

$$\text{set}(x_2) = \{\text{set}(q') \mid A_2 \in \mathcal{L}(q')\}$$

Suppose towards a contradiction that  $\text{set}(x_1) = \text{set}(x_2)$ . Hence,  $\text{set}(x_1) \subseteq \text{set}(x_2)$  and  $\text{set}(x_2) \subseteq \text{set}(x_1)$ . Since  $\text{set}(x_1) \subseteq \text{set}(x_2)$ , for all  $q$  such that  $A_1 \in \mathcal{L}(q)$  there exists  $q'$  such that  $A_2 \in \mathcal{L}(q')$  and  $\text{set}(q) = \text{set}(q')$ .

It follows from the definition of  $\prec$  that  $q \prec x_1$  and  $q' \prec x_2$ ,

By induction hypothesis, if  $q \neq q'$  then  $\text{set}(q) \neq \text{set}(q')$ . Since  $\text{set}(q) = \text{set}(q')$ , we actually have that  $q = q'$ .

Thus,  $\{q \mid A_1 \in \mathcal{L}(q)\} \subseteq \{q' \mid A_2 \in \mathcal{L}(q')\}$ , which means that:

$$A_1^{\mathcal{I}} \subseteq A_2^{\mathcal{I}} \quad (6)$$

Analogously, from  $\text{set}(x_2) \subseteq \text{set}(x_1)$  we can prove that

$$A_2^{\mathcal{I}} \subseteq A_1^{\mathcal{I}} \quad (7)$$

It follows from (6) and (7) that  $A_2^{\mathcal{I}} = A_1^{\mathcal{I}}$ . Then, applying Lemma 10 we have that  $x_1 = x_2$  which contradicts the fact that  $x_1$  and  $x_2$  are different.

Hence,  $\text{set}(x_1) \neq \text{set}(x_2)$ .

In this way we can conclude that  $\text{set}$  is injective, so it is a bijection and thus  $\mathcal{I}$  and  $\mathcal{I}'$  are isomorphic interpretations.

3. We prove that  $\mathcal{I}'$  is a model of  $\mathcal{M}$ . Suppose  $a =_{\mathcal{M}} A \in \mathcal{M}$ . From the definition of  $\mathcal{I}'$ , we know that:

$$a^{\mathcal{I}'} = \text{set}(a^{\mathcal{I}}).$$

Applying the definition of  $\text{set}$ , we have that:

$$a^{\mathcal{I}'} = \{\text{set}(y) \mid A \in \mathcal{L}(y)\}$$

Finally, applying the Definition 25 we have

$$a^{\mathcal{I}'} = \{\text{set}(y) \mid y \in A^{\mathcal{I}}\}$$

which is the definition of  $A^{\mathcal{I}'}$

Thus  $\mathcal{I}' \models a =_{\mathcal{M}} A$  for all  $a =_{\mathcal{M}} A \in \mathcal{M}$ , then  $\mathcal{I}'$  is a model of  $\mathcal{M}$ .

**Lemma 11.** *If  $(\mathcal{T}, \mathcal{A}[a/b], \mathcal{M}[a/b])$  is satisfiable,  $\{a, b\} \subseteq \text{dom}(\mathcal{M}) \cup \text{dom}(\mathcal{A})$ , then  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  is satisfiable.*

*Proof.* Suppose  $(\mathcal{T}, \mathcal{A}[a/b], \mathcal{M}[a/b])$  is satisfiable. Then, there exists  $\mathcal{I}$  such that  $\mathcal{I} \models \mathcal{A}[a/b]$ ,  $\mathcal{I} \models \mathcal{M}[a/b]$  and  $\mathcal{I} \models \bigwedge \mathcal{T} \equiv \top$ .

We define a new interpretation  $\mathcal{I}_1$  that extends  $\mathcal{I}$  by adding  $b^{\mathcal{I}_1} = a^{\mathcal{I}}$ .

- Since  $\mathcal{I} \models \bigwedge \mathcal{T} \equiv \top$ , we obviously have that

$$\mathcal{I}_1 \models \bigwedge \mathcal{T} \equiv \top. \quad (8)$$

- The Abox  $\mathcal{A}[a/b]$  is obtained from  $\mathcal{A}$  by replacing all occurrences of  $b$  by  $a$ . Let  $S \in \mathcal{A}$ . We will prove that  $\mathcal{I}_1 \models S$ .

There are several possibilities:

1. Suppose that  $S \in \mathcal{A}$  does not contain  $b$ . Then, it is easy to see that  $\mathcal{I}_1 \models S$  because  $S$  and  $S[a/b]$  are exactly the same statement and  $\mathcal{I} \models S$ .
2. Suppose that  $S$  is  $C(b)$ . Then,  $S[a/b]$  is  $C(a)$ . Hence,  $b^{\mathcal{I}_1} = a^{\mathcal{I}} \in C^{\mathcal{I}} = C^{\mathcal{I}_1}$  since  $\mathcal{I} \models C(a)$ . So  $\mathcal{I}_1 \models C(b)$ .
3. Suppose that  $S$  is  $a = b$  or  $b = a$ . Since  $b^{\mathcal{I}_1} = a^{\mathcal{I}}$ ,  $\mathcal{I}_1 \models a = b$  and  $\mathcal{I}_1 \models b = a$ .
4. Suppose that  $S$  is  $b = c$  and  $c$  is not  $a$ . Then,  $S[a/b]$  is  $a = c$  and  $\mathcal{I} \models a = c$ . Since  $b^{\mathcal{I}_1} = a^{\mathcal{I}} = c^{\mathcal{I}}$ ,  $\mathcal{I}_1 \models b = c$ .
5. Suppose that  $S$  is  $c = b$  and  $c$  is not  $a$ . This case is similar to the previous one.
6. Suppose that  $S$  is  $b \neq c$  and  $c$  is not  $a$ . Then,  $S[a/b]$  is  $a \neq c$  and  $\mathcal{I} \models a \neq c$ . Since  $b^{\mathcal{I}_1} = a^{\mathcal{I}} \neq c^{\mathcal{I}}$ ,  $\mathcal{I}_1 \models b \neq c$ .
7. Suppose that  $S$  is  $c \neq b$  and  $c$  is not  $a$ . This case is similar to the previous one.
8. Suppose that  $S$  is  $a \neq b$  or  $b \neq a$ . But then  $S[a/b]$  is  $a \neq a$ . This case is not possible because  $a \neq a$  is not satisfiable.

Hence, we have just proved that

$$\mathcal{I}_1 \models \mathcal{A}. \quad (9)$$

- Let  $S \in \mathcal{M}$ . We will prove that  $\mathcal{I}_1 \models S$ . There are two possibilities:
  1. Suppose  $S$  does not contain  $b$ . Then, it is easy to see that  $\mathcal{I}_1 \models S$ .
  2. Suppose  $S$  is  $b =_{\text{m}} A \in \mathcal{M}$ . Then  $S[a/b]$  is  $a =_{\text{m}} A$  and  $\mathcal{I} \models a =_{\text{m}} A$ . Thus,  $b^{\mathcal{I}_1} = a^{\mathcal{I}} = A^{\mathcal{I}} = A^{\mathcal{I}_1}$  and  $\mathcal{I}_1 \models b =_{\text{m}} A$ .

We conclude that

$$\mathcal{I}_1 \models \mathcal{M}. \quad (10)$$

From (8), (9) and (10) we can affirm that  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  is satisfiable.

**Lemma 12.** *Suppose there is an edge from a base node  $v$  to a base node  $w$  in an and-or graph  $\mathbb{G}$ . If the label of  $w$  is satisfiable, so is the label of  $v$ .*

*Proof.* As  $w$  is a successor of  $v$  in and-or graph  $\mathbb{G}$  and  $v$  is a base node, the label of  $w$  is obtained from the application of some rule  $r$  of the Figure 7 to the label of  $v$ . Given that  $w$  is a base node,  $r$  must be one of :  $(\sqcap')$ ,  $(\sqcup')$ ,  $(\forall)$ ,  $(\text{close})$ ,  $(=)$  or  $(\neq)$ . We will analyse each one of these cases.

$\{(\sqcap'), (\sqcup'), (\forall)\}$ -**rules** - The labels of  $v$  and  $w$  are of the form:  $(\mathcal{T}, \mathcal{A}_v, \mathcal{M})$  and  $(\mathcal{T}, \mathcal{A}_w, \mathcal{M})$  respectively. If  $(\mathcal{T}, \mathcal{A}_w, \mathcal{M})$  is satisfiable, so is  $(\mathcal{T}, \mathcal{A}_v, \mathcal{M})$  since they have the same TBox, MBox and  $\mathcal{A}_v$  is strictly contained in  $\mathcal{A}_w$ .

$(\text{close})$ -**rule** - If  $(\mathcal{T}, \mathcal{A}_v, \mathcal{M})$  is the label for  $v$ , there are two possibilities for the label of  $w$ :

1. Suppose the label is  $(\mathcal{T}, \mathcal{A}_v \cup \{a \neq b\}, \mathcal{M})$ . As in the case above if  $(\mathcal{T}, \mathcal{A}_v \cup \{a \neq b\}, \mathcal{M})$  is satisfiable, so is  $(\mathcal{T}, \mathcal{A}_v, \mathcal{M})$ .

2. Suppose the label is  $(\mathcal{T}, \mathcal{A}_v[a/b], \mathcal{M}[a/b])$ . This case follows from Lemma 11.

$(\neq)$ -**rule** - If the label of  $v$  is  $(\mathcal{T}, \mathcal{A}_v \cup \{a \neq b\}, \mathcal{M} \cup \{a =_m A, b =_m B\})$  then the label of  $w$  is of the form:  $(\mathcal{T}, \mathcal{A}_w, \mathcal{M} \cup \{a =_m A, b =_m B\})$ .

So if  $(\mathcal{T}, \mathcal{A}_w, \mathcal{M} \cup \{a =_m A, b =_m B\})$  is satisfiable, so is the label of  $v$  since they have the same TBox, MBox and  $\mathcal{A}_v \cup \{a \neq b\}$  is strictly contained in  $\mathcal{A}_w$ .

$(=)$ -**rule** - The label of  $v$  is of the form:  $(\mathcal{T}_v, \mathcal{A}_v, \mathcal{M} \cup \{a =_m A, a =_m B\})$ , the label of  $w$  is  $(\mathcal{T}_w, \mathcal{A}_w, \mathcal{M} \cup \{a =_m A\})$  where  $\mathcal{T}_w = \mathcal{T}_v \cup \{(A \sqcup \neg B) \sqcap (B \sqcup \neg A)\}$  and  $\mathcal{A}_w = \mathcal{A}_v \cup \{((A \sqcup \neg B) \sqcap (B \sqcup \neg A))(d) \mid d \in \text{dom}(\mathcal{A}_v) \cup \text{dom}(\mathcal{M}) \cup \{a\}\}$ .

Since the label of  $w$  is satisfiable, there exists  $\mathcal{I}$  such that:

1.  $\mathcal{I}$  validates all the concepts in  $\mathcal{T}_w$ , so  $\mathcal{I}$  validates all the concepts in  $\mathcal{T}_v$  and  $(A \sqcup \neg B) \sqcap (B \sqcup \neg A)$ .
2.  $\mathcal{I} \models \mathcal{A}_w$ , so  $\mathcal{I} \models \mathcal{A}_v \cup \{((A \sqcup \neg B) \sqcap (B \sqcup \neg A))(d) \mid d \in \text{dom}(\mathcal{A}_v) \cup \text{dom}(\mathcal{M}) \cup \{a\}\}$ , then  $\mathcal{I} \models \mathcal{A}_v$ .
3.  $\mathcal{I} \models \mathcal{M} \cup \{a =_m A\}$ , then  $\mathcal{I} \models \mathcal{M}$  and  $\mathcal{I} \models a =_m A$

Since  $\mathcal{I}$  validates  $(A \sqcup \neg B) \sqcap (B \sqcup \neg A)$  we have that:  $((A \sqcup \neg B) \sqcap (B \sqcup \neg A))^{\mathcal{I}} = \Delta^{\mathcal{I}}$ .

Applying the definition of interpretation we have that

$$(A^{\mathcal{I}} \cup (\Delta \setminus B^{\mathcal{I}})) \cap (B^{\mathcal{I}} \cup (\Delta \setminus A^{\mathcal{I}})) = \Delta^{\mathcal{I}}, \text{ so } (A^{\mathcal{I}} \cup (\Delta \setminus B^{\mathcal{I}})) = \Delta^{\mathcal{I}} \text{ and } (B^{\mathcal{I}} \cup (\Delta \setminus A^{\mathcal{I}})) = \Delta^{\mathcal{I}}.$$

Consequently,  $B^{\mathcal{I}} \subseteq A^{\mathcal{I}}$  and  $A^{\mathcal{I}} \subseteq B^{\mathcal{I}}$ , which it means that

$$A^{\mathcal{I}} = B^{\mathcal{I}} \tag{11}$$

Since  $\mathcal{I} \models a =_m A$  we have that:

$$a^{\mathcal{I}} = A^{\mathcal{I}} \tag{12}$$

It follows from (11) and (12) that  $a^{\mathcal{I}} = B^{\mathcal{I}}$ , so  $\mathcal{I} \models a =_m B$ .

Hence, we have proved all the necessary conditions for the satisfiability of  $(\mathcal{T}_v, \mathcal{A}_v, \mathcal{M} \cup \{a =_m A, a =_m B\})$ .

**Definition 30 (Saturation path).** Let  $\mathbb{G}$  be an and-or graph with a consistent marking  $\mathbb{G}'$  and let  $v$  be a node of  $\mathbb{G}'$ . A saturation path of  $v$  w.r.t.  $\mathbb{G}'$  is a finite sequence  $v_0 = v, v_1 \dots v_k$  of nodes of  $\mathbb{G}'$ , with  $k \geq 0$ , such that, for every  $0 \leq i < k$ ,  $v_i$  is an or-node and  $(v_i, v_{i+1})$  is an edge of  $\mathbb{G}'$ , and  $v_k$  is an and-node.

---

**Algorithm for building a saturated  $\mathbf{R}$ -structure**


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**Input:**  $\mathbb{G}'$  a consistent marking of and-or graph w.r.t.  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  such that  $\mathcal{K}$  satisfies the hypotheses of Theorem 8.

**Output:** A  $\mathbf{R}$ -structure  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  saturated for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ .

1. Let  $v_0$  be the root of  $\mathbb{G}'$ 
  - $\Delta := \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$
  - For each  $a \in \Delta$  do
    - $\mathcal{L}(a) := \{C \mid C(a) \in \mathcal{A}\}$
    - mark  $a$  as unresolved
    - $f(a) = v_0$
  - end-for
  - For each role name  $R$  do
    - $\mathcal{E}(R) := \{(a, b) \mid R(a, b) \in \mathcal{A}\}$
  - end-for
2. While  $\Delta$  contains unresolved elements, select one of them:  $x$  and do
  - For each  $\exists R.C \in \mathcal{L}(x)$  do
    - $u := f(x)$
    - Let  $w_0$  the node of  $\mathbb{G}'$  such the edge  $(u, w_0)$  is labeled by  $\exists R.C$
    - Let  $w_0 \dots w_h$  be the saturation path of  $w_0$  w.r.t.  $\mathbb{G}'$ .
    - Let  $\mathcal{Y} = \bigcup_{i=0}^h \mathcal{X}_i$  where the labels of  $w_0 \dots w_h$  are  $(\mathcal{T}, \mathcal{X}_0), \dots, (\mathcal{T}, \mathcal{X}_h)$ .
    - if there is no  $y \in \Delta$  such that  $\mathcal{L}(y) = \mathcal{Y}$  then
      - \* create a new variable  $y$  and add  $y$  to  $\Delta$
      - \*  $\mathcal{L}(y) := \mathcal{Y}$
      - \* mark  $y$  as unresolved
      - \*  $f(y) := w_h$
    - end-if
    - Add  $(x, y)$  to  $\mathcal{E}(R)$
  - end-for
  - Mark  $x$  as resolved
- end-while

---

**Table 1.** Algorithm for building a saturated  $\mathbf{R}$ -structure



Since the end-node is an and-node, all nodes of  $\mathbb{G}'$  have a saturation path.

The domain  $\Delta$  of the **R**-structure obtained by applying the algorithm of Table 1, consists of elements that also belong to  $\text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ , which we call *individuals*, and elements that are created to meet the conditions of the knowledge base given by the existential restrictions, which we call *variables*.

The function  $f$  in the algorithm of Table 1 maps elements in  $\Delta$  to an and-node of the graph  $\mathbb{G}'$ .

1. If  $a$  is an individual from the knowledge base then  $f(a) = v_0$ . The individual  $a$  comes from the unique base node which is the initial node  $v_0$ . In this case, the base node is also an and-node because no base rule is applicable to the knowledge base except for (trans').
2. If  $y$  is a variable (a newly created individual), then  $f(y) = w_h$ . The individual  $y$  comes from the last variable node  $w_h$  in the saturated path which is an and-node.

**Theorem 8 (Correctness of the Algorithm of Table 1).** *Let  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  be a knowledge base in  $\mathcal{ALCM}$  in negation normal form such that*

1.  $\mathcal{A}$  does not contain equalities,
2. no base rule is applicable to  $\mathcal{K}$  except for the (trans')-rule,
3.  $C(d) \in \mathcal{A}$  for all  $C \in \mathcal{T}$  and  $d \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .

*Suppose  $\mathbb{G}$  is an and-or graph for  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  with a consistent marking  $\mathbb{G}'$ . Then, the structure  $\mathbb{S} = (\Delta, \mathcal{L}, \mathcal{E})$  obtained by applying the algorithm of Table 1 is a saturated **R**-structure for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ .*

*Proof.* We first prove that  $\mathbb{S}$  is a saturated **R**-structure for  $\mathcal{T}$ . For this, we need to prove the six conditions of Definition 24. Suppose  $x, y \in \Delta$ ,  $R \in \mathbf{R}$  and concepts  $C, C_1, C_2, D$ .

1. Suppose towards a contradiction that there is  $x \in \Delta$  such that  $\{\neg C, C\} \subseteq \mathcal{L}(x)$ . Since all concepts are in NNF,  $C$  is an atomic concept  $B$ , so  $\{\neg B, B\} \subseteq \mathcal{L}(x)$ . We divide in two cases:
  - (a) Suppose  $x \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .  
By construction,  $\{\neg B(x), B(x)\} \subseteq \mathcal{A}$ . But then we can apply the  $(\perp_1)$ -rule which it is not possible by the second hypothesis for  $\mathcal{K}$ , namely no base rule is applicable to  $\mathcal{K}$  except for the (trans')-rule.
  - (b) Suppose  $x \notin \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .  
By construction,  $\mathcal{L}(x) = \bigcup_{i=0}^h \mathcal{X}_i$  where  $(\mathcal{T}, \mathcal{X}_0), \dots, (\mathcal{T}, \mathcal{X}_h)$  are the labels of the saturation path  $w_0, \dots, w_h$ ,  $f(x) = w_h$ . It is not possible that each concept is on a different node since  $w_0, \dots, w_h$  is a saturation path and no rules eliminate an atomic concept of the label of a node. Then,  $\{\neg B, B\} \subseteq \mathcal{X}_i$  for some  $i$  such that  $1 \leq i \leq h$ . By Definition 20, the  $(\perp)$ -rule has higher priority than any of the other rules. This means that  $w_i$  should be the last node in the path, i.e.  $i = h$ . But then,  $w_h$  would be an absurdity node which contradicts the  $\mathbb{G}'$  is a consistent marking.

2. Suppose  $C_1 \sqcap C_2 \in \mathcal{L}(x)$ . We split in two cases:

(a) Suppose  $x \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .

By construction,  $C_1 \sqcap C_2 \in \mathcal{L}(x)$  if  $(C_1 \sqcap C_2)(x) \in \mathcal{A}$ . Then we have that  $\{C_1(x), C_2(x)\} \subseteq \mathcal{A}$  because the  $(\sqcap')$ -rule is not applicable to  $\mathcal{A}$ . Hence,  $\{C_1, C_2\} \subseteq \mathcal{L}(x)$ .

(b) Suppose  $x \notin \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .

By construction,  $\mathcal{L}(x) = \bigcup_{i=0}^h \mathcal{X}_i$  where  $f(x) = w_h$  and the labels of the saturation path  $w_0, \dots, w_h$  are  $(\mathcal{T}, \mathcal{X}_0), \dots, (\mathcal{T}, \mathcal{X}_h)$ . Hence, there exists  $1 \leq i \leq h$  such that  $C_1 \sqcap C_2 \in \mathcal{X}_i$ . We have that  $C_1 \sqcap C_2 \notin \mathcal{X}_h$  because  $w_h$  is an and-node and the  $(\sqcap)$ -rule is not applicable to  $(\mathcal{T}, \mathcal{X}_h)$ . Hence, there is an  $i < h$  such that  $C_1 \sqcap C_2 \in \mathcal{X}_i$ ,  $C_1 \sqcap C_2 \notin \mathcal{X}_{i+1}$  and  $C_1, C_2 \in \mathcal{X}_{i+1}$ . Hence,  $C_1, C_2 \in \mathcal{X}_{i+1} \subseteq \mathcal{L}(x)$ .

3. Suppose  $C_1 \sqcup C_2 \in \mathcal{L}(x)$ . We divide in two cases:

(a) Suppose  $x \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .

By construction,  $C_1 \sqcup C_2 \in \mathcal{L}(x)$  if  $(C_1 \sqcup C_2)(x) \in \mathcal{A}$ .

Then we have that  $C_1(x) \in \mathcal{A}$  or  $C_2(x) \in \mathcal{A}$  because the  $(\sqcup')$ -rule is not applicable to  $\mathcal{A}$ . Hence,  $C_1 \in \mathcal{L}(x)$  or  $C_2 \in \mathcal{L}(x)$ .

(b) Suppose  $x \notin \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .

By construction,  $\mathcal{L}(x) = \bigcup_{i=0}^h \mathcal{X}_i$  where  $f(x) = w_h$  and the labels of the saturation path  $w_0, \dots, w_h$  are  $(\mathcal{T}, \mathcal{X}_0), \dots, (\mathcal{T}, \mathcal{X}_h)$ .

$C_1 \sqcup C_2 \notin \mathcal{X}_h$  because  $w_h$  is and-node and the  $(\sqcup)$ -rule is not applicable to  $(\mathcal{T}, \mathcal{X}_h)$ . Hence, there exists  $1 \leq i < h$  such that  $C_1 \sqcup C_2 \in \mathcal{X}_i$ ,  $C_1 \sqcup C_2 \notin \mathcal{X}_{i+1}$  and  $C_1 \in \mathcal{X}_{i+1}$  or  $C_2 \in \mathcal{X}_{i+1}$ . So  $C_1 \in \mathcal{L}(x)$  or  $C_2 \in \mathcal{L}(x)$ .

4. Assume  $x \in \Delta, \forall R.D \in \mathcal{L}(x)$  and  $(x, y) \in \mathcal{E}(R)$ , we show that  $D \in \mathcal{L}(y)$ .

(a) Suppose  $x \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .

By construction,  $\forall R.D(x) \in \mathcal{A}$ .

– Suppose  $y \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .

Then  $R(x, y) \in \mathcal{A}$  since  $(x, y) \in \mathcal{E}(R)$ . Then we have that  $D(y) \in \mathcal{A}$  because the  $(\forall)$ -rule is not applicable to  $\mathcal{A}$ . Hence,  $D \in \mathcal{L}(y)$ .

– Suppose  $y \notin \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .

By construction,  $\mathcal{L}(y) = \bigcup_{i=0}^h \mathcal{X}_i$  where the labels of the saturation path  $w_0, \dots, w_h$  are  $(\mathcal{T}, \mathcal{X}_0), \dots, (\mathcal{T}, \mathcal{X}_h)$ . Since  $\forall R.D(x) \in \mathcal{A}$ , it belongs to the label of  $v_0$ . Since  $R(x, y)$  cannot belong to  $\mathcal{A}$  because  $y \notin \text{dom}(\mathcal{A})$ , we have that  $\exists R.C(x) \in \mathcal{A}$  for some concept  $C$ . By application of the  $(\text{trans}')$ -rule we know that  $D$  belongs to the label of  $w_0$  and hence also to  $\mathcal{L}(y)$ .

(b) Suppose  $x \notin \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .

Then,  $y \notin \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$  because in  $\mathcal{E}$  there are no pairs  $(x, y)$  where  $x \notin \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$  and  $y \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ .

So by construction, we know that:

–  $\mathcal{L}(x) = \bigcup_{i=0}^h \mathcal{X}_i$  where the labels of the saturation path  $w_0, \dots, w_h$  are  $(\mathcal{T}, \mathcal{X}_0), \dots, (\mathcal{T}, \mathcal{X}_h)$ .

–  $\mathcal{L}(y) = \bigcup_{i=0}^p \mathcal{Z}_i$  where the labels of the saturation path  $v_0, \dots, v_p$  are  $(\mathcal{T}, \mathcal{Z}_0), \dots, (\mathcal{T}, \mathcal{Z}_p)$ .

Since  $(x, y) \in \mathcal{E}(R)$  and  $R(x, y) \notin \mathcal{A}$ , we know that  $\exists R.C_1 \in \mathcal{L}(x)$  for some  $C_1$ . Since the rules applied to the nodes  $w_0, \dots, w_{h-1}$  do not eliminate for all and exists concepts, we know that  $\{\forall R.D, \exists R.C_1\} \subseteq \mathcal{X}_h$ . The node  $v_0$  is obtained from  $w_h$  by application of the (trans)-rule.

In  $\mathbb{G}'$  there is an edge  $(w_h, v_0)$  labelled  $\exists R.C_1$ .

By the application of the (trans)-rule at the node  $w_h$  we know that  $D \in \mathcal{Z}_0$ , so  $D \in \mathcal{L}(y)$ .

5. Assume  $\exists R.C \in \mathcal{L}(x)$  for  $x \in \Delta$ .

At the step 2 of the algorithm we create (if it does not exist) a new variable  $y$  and add  $(x, y)$  to  $\mathcal{E}(R)$ . For this  $y$ ,  $\mathcal{L}(y) = \bigcup_{i=0}^h \mathcal{X}_i$  where the labels of the saturation path  $w_0, \dots, w_h$  are  $(\mathcal{T}, \mathcal{X}_0), \dots, (\mathcal{T}, \mathcal{X}_h)$ .

Let  $u = f(x)$  (the node associated with  $x$  in  $\mathbb{G}'$ ).

Then,  $w_0$  is obtained from  $u$  by applying the (trans) or (trans')-rule. So  $C \in \mathcal{X}_0$  and also  $C \in \mathcal{L}(y)$ .

6. We prove that  $\mathcal{T} \subseteq \mathcal{L}(x)$  for all  $x \in \Delta$ . We divide in cases:

- (a) Suppose  $x \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ . By initialization,

$$\mathcal{L}(x) = \{C \mid C(a) \in \mathcal{A}\}$$

Then,  $\mathcal{T} \subseteq \mathcal{L}(x)$  by the third hypothesis on  $\mathcal{K}$ .

- (b) Suppose  $x \notin \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$ . By construction,  $\mathcal{L}(x) = \bigcup_{i=0}^h \mathcal{X}_i$  where  $f(x) = w_h$  and the labels of the saturation path  $w_0, \dots, w_h$  are  $(\mathcal{T}, \mathcal{X}_0), \dots, (\mathcal{T}, \mathcal{X}_h)$ . The node  $w_0$  was obtained by application of either the (trans')-rule or (trans)-rule. Since both rules add the Tbox to  $\mathcal{X}_0$ , we have that

$$\mathcal{T} \subseteq \mathcal{X}_0 \subseteq \mathcal{L}(x)$$

We now prove that  $\mathbb{S}$  is a saturated **R**-structure for  $(\mathcal{T}, \mathcal{A})$ . For this, it only remains to prove the four conditions of Definition 26.

1. By step 1 in the algorithm of Table 1 all the individuals of  $\mathcal{A}$  belongs to  $\Delta$ .
2. It holds by initialization of  $\mathcal{L}(a)$  (step 1 in the algorithm) for each  $a \in \text{dom}(\mathcal{A}) \cup \text{dom}(\mathcal{M})$  in the first step of algorithm, so if  $C(a) \in \mathcal{A}$ , then  $C \in \mathcal{L}(a)$ .
3. It holds by initialization of  $\mathcal{E}(R)$  for each role name  $R$  (step 1 in the algorithm), so if  $R(a, b) \in \mathcal{A}$ , then  $(a, b) \in \mathcal{E}(R)$ .
4. If  $a \neq b \in \mathcal{A}$  then  $a$  and  $b$  are syntactically different, otherwise we could apply  $(\perp_2)$ -rule which it is not possible by the second hypothesis for  $\mathcal{K}$ .

We now prove that  $\mathbb{S}$  is a **R**-structure for  $\mathcal{ALCM}$ . For this, we only need to prove the four conditions of Definition 28.

1. By step 1 of the algorithm all the individuals of  $\mathcal{M}$  belongs to  $\Delta$ .

2. Suppose towards a contradiction that  $\mathbb{S}$  has circularities w.r.t. a  $\mathcal{M}$ . Then there is a sequence of meta-modelling axioms  $a_1 =_m A_1, a_2 =_m A_2 \dots a_n =_m A_n$  all in  $\mathcal{M}$  such that  $A_1 \in \mathcal{L}(a_2), A_2 \in \mathcal{L}(a_3) \dots A_n \in \mathcal{L}(a_1)$ . That is,  $A_1(a_2), A_2(a_3), \dots, A_n(a_1)$  are in  $\mathcal{A}$ , so from Definition 19 we have  $\text{circular}(\mathcal{A}, \mathcal{M})$ . But then we can apply the  $(\perp_3)$ -rule which contradicts the second hypothesis for  $\mathcal{K}$ .
3. Suppose  $a =_m A \in \mathcal{M}$  and  $a =_m B \in \mathcal{M}$ . Then,  $A = B$ . Otherwise we could apply the  $(=)$ -rule which it is not possible by the second hypothesis for  $\mathcal{K}$ .
4. Suppose  $a$  and  $b$  are syntactically different,  $a =_m A \in \mathcal{M}$  and  $b =_m B \in \mathcal{M}$ . Then there is some  $t \in \Delta$  such that  $A \sqcap \neg B \sqcup B \sqcap \neg A \in \mathcal{L}(t)$ . Otherwise we could apply the  $(\neq)$ -rule which it is not possible by the second hypothesis for  $\mathcal{K}$ .

**Theorem 9 (Completeness of the Tableau Calculus of  $\mathcal{ALCCM}$ ).** *Let  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  be a knowledge base of  $\mathcal{ALCCM}$  in negation normal form. If the and-or graph for  $\mathcal{K}$  has a consistent marking then  $\mathcal{K}$  is consistent.*

*Proof.* Suppose  $\mathbb{G}'$  is a consistent marking of the and-or graph  $\mathbb{G}$  of  $\mathcal{K}$ . Let  $v_0, v_1, \dots, v_n$  be the saturation path from the root of  $\mathbb{G}'$ . It is clear that every  $v_i$  is a base node for all  $1 \leq i \leq n - 1$ . The last node cannot be an absurdity node because  $\mathbb{G}'$  a consistent marking. Hence,  $v_n$  should be a base node as well. This means that the node  $v_i$  has label

$$(\mathcal{T}_i, \mathcal{A}_i, \mathcal{M}_i)$$

and in particular the label of  $v_0$  is

$$(\mathcal{T}_0, \mathcal{A}_0, \mathcal{M}_0)$$

where

$$\begin{aligned} \mathcal{T}_0 &:= \mathcal{T} \\ \mathcal{A}_0 &:= \mathcal{A}^* \cup \{C(a) \mid C \in \mathcal{T}, a \in \text{dom}(\mathcal{A}^*) \cup \text{dom}(\mathcal{M}^*)\} \\ \mathcal{M}_0 &:= \mathcal{M}^* \end{aligned}$$

where  $\mathcal{A}^*$  and  $\mathcal{M}^*$  are obtained from  $\mathcal{A}$  and  $\mathcal{M}$  by choosing a canonical representative  $a$  for each assertion  $a = b$  and replacing  $b$  by  $a$ .

We will apply the algorithm of Table 1 to  $(\mathcal{T}_n, \mathcal{A}_n, \mathcal{M}_n)$ . For this we need to prove that the hypotheses of Theorem 8 hold:

1. No base rule is applicable to  $(\mathcal{T}_n, \mathcal{A}_n, \mathcal{M}_n)$  except for  $(\text{trans}')$ . To prove this we divide in cases:
  - Suppose  $v_n$  is an end-node. Then, it follows trivially that no rule is applicable to the node.
  - Suppose that the rule  $(\text{trans}')$  was applied to  $v_n$  to obtain its successors in the and-or graph  $\mathbb{G}$ . It follows from Definition 20 that the  $(\text{trans}')$ -rule could be applied to  $v_n$  only if the other rules are not applicable.
2. It is easy to see that  $\mathcal{A}_n$  does not contain equalities since  $\mathcal{A}_0$  does not contain equalities.

3. We also have that  $C(d) \in \mathcal{A}$  for all  $C \in \mathcal{T}$  and  $d \in \text{dom}(\mathcal{A}_n) \cup \text{dom}(\mathcal{M}_n)$ . This property holds in the initialization and it is also preserved after applying any of the base rules. Note that the only rule we could apply in the saturation path  $v_0, \dots, v_n$  that adds new individuals is the  $(\neq)$ -rule.

It follows from Theorem 8 that there exists a saturated **R**-structure for  $(\mathcal{T}_n, \mathcal{A}_n, \mathcal{M}_n)$ . By Theorem 7,  $(\mathcal{T}_n, \mathcal{A}_n, \mathcal{M}_n)$  is satisfiable. Applying the Lemma 12 we know that  $(\mathcal{T}_0, \mathcal{A}_0, \mathcal{M}_0)$  is satisfiable and from Lemma 11 that  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$  is satisfiable too.

## 8 An ExpTime Decision Procedure for $\mathcal{ALCM}$

In this section, we prove that the complexity does not increase when we move from  $\mathcal{ALC}$  to  $\mathcal{ALCM}$ . In order to prove this, it is enough to give an algorithm for checking consistency of a knowledge base in  $\mathcal{ALCM}$  that is ExpTime.

Let  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  a knowledge base in  $\mathcal{ALCM}$  with  $\mathcal{T}$  and  $\mathcal{A}$  in negation normal form (NNF). We claim that the *Algorithm for Checking Consistency in  $\mathcal{ALCM}$*  given in Table 2 is an ExpTime (complexity-optimal) algorithm for checking consistency of  $\mathcal{K}$ . In the algorithm, a node  $u$  is a parent of a node  $v$  and  $v$  is a child of  $u$  iff the edge  $(u, v)$  is in  $\mathbb{G}$ . Note also that there is a unique node with label  $\perp$ .

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### Algorithm for Checking Consistency in $\mathcal{ALCM}$

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**Input:**  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  in negation normal form.

**Output:** *true* if  $\mathcal{K}$  is consistent, and *false* otherwise.

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1. Construct and “and-or” graph  $\mathbb{G}$  with root  $v_0$  for  $(\mathcal{T}, \mathcal{A}, \mathcal{M})$ ;
  2.  $\text{UnsatNodes} := \emptyset, U := \emptyset$ ;
  3. If  $\mathbb{G}$  contains a node  $v_\perp$  with label  $\perp$  then  
 $U := \{v_\perp\}, \text{UnsatNodes} := \{v_\perp\}$ ;  
while  $U$  is not empty do  
  remove a node  $v$  from  $U$ ;  
  for every parent  $u$  of  $v$  do  
    if  $u \notin \text{UnsatNodes}$  and ( $u$  is an and-node or  $u$  is an or-node and every child of  $u$  is in  $\text{UnsatNodes}$ ) then add  $u$  to both  $\text{UnsatNodes}$  and  $U$
  4. return *false* if  $v_0 \in \text{UnsatNodes}$ , and *true* otherwise
- 

**Table 2.** Algorithm for checking consistency in  $\mathcal{ALCM}$

Recall that a *formula* is either a concept, or an Abox-statement or an Mbox-statement. We define the *length* of a formula to be the number of its symbols, and the *size* of a finite set of formulas to be the sum of the lengths of its elements.

**Lemma 13.** *Let  $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{M})$  be an  $\mathcal{ALCM}$ -knowledge base in negation normal form,  $n$  be the size of  $\mathcal{T} \cup \mathcal{A} \cup \mathcal{M}$ , and  $\mathbb{G}$  be an and-or graph for  $\mathcal{K}$ . Then  $\mathbb{G}$  has  $O(2^{n^4})$  nodes.*

*Proof.* For each pair  $a, b$  of individuals with meta-modelling in the Mbox, the algorithm either adds a new TBox axiom, using the  $(=)$ -rule, or adds an individual that is denoted by  $d_{a,b}$ , using the  $(\neq)$ -rule. We define

$$\begin{aligned}\mathcal{T}^+ &= \mathcal{T} \cup \{(A \sqcup \neg B) \sqcap (B \sqcup \neg A) \mid a =_{\mathcal{M}} A, b =_{\mathcal{M}} B \in \mathcal{M}\} \\ \mathcal{A}^+ &= \mathcal{A} \cup \{(A \sqcap \neg B \sqcup B \sqcap \neg A)(d_{a,b}) \mid a =_{\mathcal{M}} A, b =_{\mathcal{M}} B \in \mathcal{M}\}\end{aligned}$$

The sets  $\mathcal{T}^+$  and  $\mathcal{A}^+$  have cardinality  $O(n^2)$  since

$$\#\{(a, b) \mid a, b \in \text{dom}(\mathcal{M})\} \leq n^2 \quad (13)$$

The label of each base node  $v$  of  $\mathbb{G}$  is  $(\mathcal{T}_v, \mathcal{A}_v, \mathcal{M}_v)$ . The sets  $\mathcal{T}_v$ ,  $\mathcal{A}_v$  and  $\mathcal{M}_v$  have the following upper bounds:

$$\mathcal{T}_v \subseteq \mathcal{T}^+$$

$$\begin{aligned}\mathcal{A}_v &\subseteq \{D(b) \mid C(a) \in \mathcal{A}^+, D \in \text{sc}(C) \text{ and } b \in \text{dom}(\mathcal{A}^+) \cup \text{dom}(\mathcal{M})\} \cup \\ &\quad \{C(a) \mid C \in \text{sc}(\mathcal{T}^+) \text{ and } a \in \text{dom}(\mathcal{A}^+) \cup \text{dom}(\mathcal{M})\} \cup \\ &\quad \{a \neq b \mid a \neq b \in \mathcal{A} \text{ or } a, b \in \text{dom}(\mathcal{M})\}\end{aligned}$$

$$\mathcal{M}_v \subseteq \{a =_{\mathcal{M}} A \mid a \in \text{dom}(\mathcal{M}), A \in \text{range}(\mathcal{M})\}$$

where  $\text{sc}(\mathcal{T}^+)$  denotes the image of  $\mathcal{T}^+$  under  $\text{sc}$ , i.e.  $\text{sc}(\mathcal{T}^+) = \{\text{sc}(C) \mid C \in \mathcal{T}^+\}$ .

Recall  $\text{sc}(C)$  gives the set of subconcepts of  $C$ .

The upper bound for  $\mathcal{A}_v$  is the union of three sets:

$$\begin{aligned}&\{D(b) \mid C(a) \in \mathcal{A}^+, D \in \text{sc}(C) \text{ and } b \in \text{dom}(\mathcal{A}^+) \cup \text{dom}(\mathcal{M})\} \cup \\ &\{C(a) \mid C \in \text{sc}(\mathcal{T}^+) \text{ and } a \in \text{dom}(\mathcal{A}^+) \cup \text{dom}(\mathcal{M})\} \cup \\ &\{a \neq b \mid a \neq b \in \mathcal{A} \text{ or } a, b \in \text{dom}(\mathcal{M})\}\end{aligned}$$

The first set  $\{D(b) \mid C(a) \in \mathcal{A}^+, D \in \text{sc}(C) \text{ and } b \in \text{dom}(\mathcal{A}^+) \cup \text{dom}(\mathcal{M})\}$  includes the axioms  $C(b)$  that are obtained from replacing  $a$  by  $b$  in  $C(a) \in \mathcal{A}$ .

The cardinality of the first set has  $O(n^4) = O(n^2) \times O(n^2)$ . This is because we are combining  $D$ 's with  $b$ 's. The number of  $D$ 's as well as the number of  $b$ 's have  $O(n^2)$  since in both cases the cardinality of  $\mathcal{A}^+$  is the predominant one.

Similarly, the cardinality of the second set has  $O(n^4) = O(n^2) \times O(n^2)$ . This is because we are combining  $C$ 's with  $a$ 's. There are as many  $C$ 's as elements in  $\text{sc}(\mathcal{T}^+)$  and the latter has  $O(n^2)$ . The number of  $a$ 's has the same order as the cardinality of  $\mathcal{A}^+$  which is  $O(n^2)$ .

It follows from (13) that the third set has cardinality  $O(n^2)$ .

Hence, the cardinality of the upper bound for  $\mathcal{A}_v$  which is the union of these three sets has  $O(n^4)$ .

The number of base nodes has the following order:

$$O(2^{n^2}) \times O(2^{n^4}) \times O(2^{n^2}) = O(2^{n^4})$$

which is the multiplication of the number of subsets of the upper bounds for  $\mathcal{T}_v$ ,  $\mathcal{A}_v$  and  $\mathcal{M}_v$ .

The label of each variable node  $w$  of  $\mathbb{G}$  is  $(\mathcal{T}_w, \mathcal{X}_w)$  where

$$\mathcal{T}_w \subseteq \mathcal{T}^+$$

$$\mathcal{X}_w \subseteq \{D \mid C(a) \in \mathcal{A}^+ \text{ and } D \in \text{sc}(C)\} \cup \{C \mid C \in \text{sc}(\mathcal{T}^+)\}$$

The cardinality of the bound for  $\mathcal{X}_w$  has  $O(n^2)$ . Hence, the number of variables nodes are

$$O(2^{n^2}) \times O(2^{n^2}) = O(2^{n^2})$$

which is the multiplication of the number of subsets of the upper bounds for  $\mathcal{T}_w$  and  $\mathcal{X}_w$ .

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### Algorithm for Checking Circularities

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**Input:** an Abox  $\mathcal{A}$  and an Mbox  $\mathcal{M}$ .

**Output:** *true* if there exists a circularity in  $\mathcal{A}$  w.r.t.  $\mathcal{M}$  and *false* otherwise.

1. Given the Abox  $\mathcal{A}$  and Mbox  $\mathcal{M}$ , construct a directed graph as follows.
  - (a) The nodes are the elements in  $\text{dom}(\mathcal{M})$ .
  - (b) There is an edge from  $a$  to  $b$  if  $B(a) \in \mathcal{A}$  and  $b =_{\mathcal{M}} B$ .
2. Check if there is a cycle in the graph constructed in the previous part using some well-known algorithm for cycle detection, e.g. Chapter 4 of [25].

**Table 3.** Algorithm for Checking Circularities

The construction of the and-or graph needs to check if each node has circularities or not. We show an algorithm for checking circularities in Table 3. For example, consider the Mbox

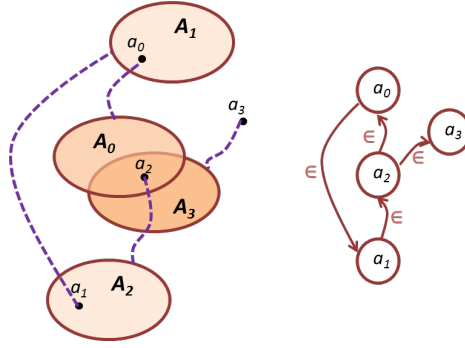
$$a_0 =_{\mathcal{M}} A_0 \quad a_1 =_{\mathcal{M}} A_1 \quad a_2 =_{\mathcal{M}} A_2 \quad a_3 =_{\mathcal{M}} A_3$$

and the Abox

$$A_1(a_0) \quad A_0(a_2) \quad A_3(a_2) \quad A_2(a_1)$$

We construct the graph illustrated in Figure 9, whose nodes are  $a_0, a_1, a_2$  and  $a_3$ . In this graph, there is an edge from the node  $a_i$  to  $a_j$  if  $(a_i)^{\mathcal{I}} \in (a_j)^{\mathcal{I}}$  for a model  $\mathcal{I}$ . In other words, the edges represent the membership relation  $\in$ .

**Lemma 14.** *The Algorithm for Checking Consistency in  $\mathcal{ALCM}$  of Table 2 terminates and computes the set  $\text{UnsatNodes}$  in  $O(2^{n^4})$  steps where  $n$  is the size of  $\mathcal{T} \cup \mathcal{A} \cup \mathcal{M}$ .*



**Fig. 9.** Knowledge base with cycles and associated directed graph

*Proof.* Each node is processed in  $O(n)$  since checking for clashing and circularities take  $O(n)$ . For checking circularities, we need to detect cycles in a graph (the edges represent the membership relation  $\in$ ) which takes  $O(n)$  [25] (see Table 3). Lemma 13 guarantees that the and-or graph  $\mathbb{G}$  can be built in  $O(2^{n^4})$ . Every node put into  $U$  is also put into  $\text{UnsatNodes}$ , but once a node is in  $\text{UnsatNodes}$ , it never leaves  $\text{UnsatNodes}$  and cannot be put back into  $U$ . Each iteration of the “while” removes one member of  $U$ . Since the number of nodes in  $\mathbb{G}$  is  $O(2^{n^4})$ , this means that after at most  $O(2^{n^4})$  iterations,  $U$  become empty. Each iteration is done in  $O(2^{n^4})$  steps. Hence the algorithm terminates after  $O(2^{n^4})$  steps.

**Theorem 10.** *The algorithm of Table 2 is an ExpTime decision procedure for checking consistency of a knowledge base in  $\mathcal{ALCM}$ .*

*Proof.* The proof of correctness is the same as of [20, Theorem 5.3]. We have to use Theorem 2. Complexity follows from Lemma 14.

We can now show the main new result of this paper:

**Corollary 1 (Complexity of  $\mathcal{ALCM}$ ).** *Consistency of a (general) knowledge base in  $\mathcal{ALCM}$  is ExpTime-complete.*

Hardness follows from the corresponding result for  $\mathcal{ALC}$  (see Theorem 1). A matching upper bound for  $\mathcal{ALCM}$  is given by the algorithm of Table 2 which by Theorem 10 is ExpTime.

## 9 Related Work

We made several changes to the ExpTime tableau algorithm for  $\mathcal{ALC}$  by Nguyen and Szalas to accommodate meta-modelling [15]. First of all, we added some rules for dealing with the equalities and inequalities that need to be transferred from the Tbox to the Abox and vice versa. There is also a new rule that returns inconsistency in case a circularity is found. This new rule is key for our approach to meta-modelling and ensures



that the domain of the interpretation is well-founded. Since our tableau algorithm has the peculiarity of changing the TBox, the Tbox (and also the Mbox) have to be stored in the labels of the and-or graph, a fact that was not necessary in the simpler Tableau Calculus for  $\mathcal{ALC}$  of Nguyen and Szalas [15].

In the literature of Description Logic, there are other approaches to meta-modelling [7,8,9,10,11,12,13,14]. The approaches which define fixed layers or levels of meta-modelling [8,10,12,13] impose a very strong limitation to the ontology engineer. Our approach allows the user to have any number of levels or layers (meta-concepts, meta meta-concepts and so on). Besides the benefits of not having to know the layer of each concept and having the flexibility of mixing different layers, there is a more pragmatic advantage which arises from ontology engineering. In a real scenario of evolving ontologies, that need to be integrated, not all individuals of a given concept need to have meta-modelling and hence, they do not have to belong to the same level in the hierarchy.

The key feature in our semantics is to interpret  $a$  and  $A$  as the same object when  $a$  and  $A$  are connected through meta-modelling, i.e., if  $a =_m A$  then  $a^{\mathcal{I}} = A^{\mathcal{I}}$ . This allows us to detect inconsistencies in the ontologies which is not possible under the Hilog semantics [7,11,12,13,14,26]. Our semantics also requires that the domain of the interpretation be a well-founded set. A domain such as  $\Delta^{\mathcal{I}} = \{X\}$  where  $X = \{X\}$  is a set that belongs to itself, it cannot represent any real object from our usual applications in Semantic Web.

## 10 Conclusions and Future Work

The ExpTime algorithm for  $\mathcal{ALCM}$  presented in this paper can be optimized in several ways. Instead of constructing first the and-or graph and then checks whether the graph contains a consistent marking, we can do these two tasks simultaneously [20]. Adding  $(A \sqcup \neg B) \sqcap (B \sqcup \neg A)$  to the Tbox is not efficient since it generates too many expansions with or-branching. We can instead add  $A \equiv B$  and apply optimizing techniques of lazy unfolding [23].

We plan to extend this algorithm to include other logical constructors such as cardinality restrictions, role hierarchies and nominals [27,28].

It is also possible to show Pspace-completeness for  $\mathcal{ALCM}$  under certain conditions of unfoldable Tboxes. The details of this proof will appear in a separate report.

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